CONSTITUTIONAL IMPLEMENTATION OF AFFIRMATIVE ACTION POLICIES IN INDIA

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ABSTRACT. India is home to a comprehensive affirmative action program that reserves a fraction of positions at governmental institutions for various disadvantaged groups. While there is a Supreme Court-endorsed mechanism to implement these reservation policies when all positions are identical, courts have refrained from endorsing explicit mechanisms when positions are heterogeneous. This lacunae has resulted in widespread adoption of unconstitutional mechanisms, countless lawsuits, and inconsistent court rulings. Formulating mandates in the landmark Supreme Court judgment *Saurav Yadav* (2020) as technical axioms, we show that the 2SMH-DA mechanism is uniquely suited to overcome these challenges.

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1. Introduction

The landmark Supreme Court judgment *Indra Sawhney and others v. Union of India (1992)*[1] formulates a number of affirmative action provisions built into the Constitution of India. Allocation of seats in the country's legislative bodies, public employment, and publicly funded educational institutions are governed by the principles outlined in this ruling. Equity is embedded within a merit-based system under these principles through two types of affirmative action policies called *vertical reservations* (VR) and *horizontal reservations* (HR). In order to operationalize these principles in practical applications, merit is typically determined with entrance exams (or the number of votes for legislative seat allocation), and the two affirmative action policies are implemented by reserving a fraction of positions for each of a number of protected groups. Of the two policies, the VR policy is envisioned as the main and higher-level affirmative action policy, and thereby it is mandated to be implemented on an "over-and-above" basis. This means that if a member of a VR-protected class is "entitled" to receive an open (or unreserved) position based on merit, then she must be awarded an open position and not use up a reserved position. Therefore, the protected positions are exclusively awarded to eligible individuals who do not merit an open position under the VR policy. This primary reservation policy has largely been intended for historically oppressed classes, most notably *Scheduled Castes (SC)*, *Scheduled Tribes (ST)*, and *Other Backward Classes (OBC)*. The HR policy, on the other hand, is envisioned as a secondary and lower-level affirmative action policy, and thereby it is mandated to be implemented on a "minimum guarantee" basis. This means that any position awarded to a member of an HR-protected group counts towards HR protections. Also known as the *interlocking reservations* in India, the HR policy is typically implemented separately within open positions and each category of VR-protected positions.

As they are stated in *Indra Sawhney (1992)*, the formulation of VR and HR policies becomes airtight under the following three conditions:

(1) *Homogeneity*: All positions are identical.
(2) *Stand-alone implementation*: Only one of the reservation policies is implemented.
(3) *Non-overlapping protected groups*: No individual belongs to multiple protected groups.[2]

However, this landmark judgment, which serves as the primary reference for India's reservation system, has not provided detailed guidance when any combination of the three conditions fail, a scenario that holds for a vast majority of field applications in the country.

[1]The ruling, widely known as the *Mandal Commission Case* and considered one of the most important judgments of the Supreme Court of India, is available at `https://indiankanoon.org/doc/1363234/` (retrieved on 02/20/2022).

[2]In India VR-protected groups do not overlap with each other. However, HR-protected groups both overlap with VR-protected groups and sometimes overlap with other HR-protected groups.

For the last three decades this lack of precision not only resulted in adoption of numerous flawed allocation mechanisms in India, but also in countless litigations and inconsistent rulings at all three levels of the country's judicial system.

In an effort to eliminate the above-described imprecision for field applications where positions are identical but the other two restrictive conditions are dropped (i.e., when the two policies are implemented together and protected groups potentially overlap), a procedure was formulated and enforced countrywide in a subsequent judgement *Anil Kumar Gupta, Etc vs State Of Uttar Pradesh & Ors (1995)* of the Supreme Court.[3] However, a key flaw in this procedure has further aggravated the crisis for the next twenty five years. We documented the crisis due to the flawed Supreme Court procedure (henceforth the *SCI-AKG choice rule*) in Sönmez and Yenmez (2021), and formulated a remedy through a mechanism we refer to as the *two-step meritorious horizontal (2SMH) choice rule*. Parallel to our analysis and policy recommendations in Sönmez and Yenmez (2021), the flawed SCI-AKG choice rule is recently rescinded and the 2SMH choice rule is endorsed[4] by a three-judge full bench[5] of the Supreme Court in *Saurav Yadav vs The State Of Uttar Pradesh (2020)*.[6]

While *Saurav Yadav (2020)* has finally resolved the legal inconsistencies and provided Indian institutions with a well-behaved mechanism when all positions are identical, to the best of our knowledge no mechanism has ever been endorsed or mandated to this date by any court in the country when the positions are heterogenous.[7] This lacuna has forced local agencies to design their own mechanisms,[8] which have subsequently been challenged in court and resulted in numerous inconsistent decisions at all three tiers of the Indian Judicial System.[9]

In this paper, we aim to fill this gap by formulating a mechanism that satisfies the core principles outlined in *Indra Sawhney (1992)* for the most general version of the problem

[3]The ruling is available at `https://indiankanoon.org/doc/1055016/` (retrieved on 12/25/2021).

[4]Strictly speaking the endorsement in *Saurav Yadav (2020)* assumes away any possible overlap in HR-protected groups. The 2SMH choice rule takes a simpler form referred to as the *two-step minimum guarantee (2SMG) choice rule* in this case.

[5]Typically, the cases filed at the Supreme Court of India is decided by two judges through what is referred to as a *division bench*. For example, *Anil Kumar Gupta (1992)* was decided by a division bench. Deviating from this norm, larger benches may be formed in some important and potentially controversial cases. A bench consisting of three or four judges is known as a *full branch* and a bench consisting of five or more judges is known as a *constitution bench*. The landmark case *Indra Sawhney (1992)* was decided by a nine-judge constitution bench of the Supreme Court.

[6]The ruling is available at `https://indiankanoon.org/doc/27820739/` (retrieved on 12/25/2021).

[7]This more general version of the problem is important in India, because, not only it covers allocation of some of the most prestigious public jobs (e.g., the Indian Administrative Service positions), but also the assignment of public college seats.

[8]To the best of our knowledge, no institution in India has been able design a mechanism that abides by the principles outlined in *Indra Sawhney (1992)* when the positions are heterogenous.

[9]See Section 5 for some of the main decisions with important flaws.

where all three restrictive conditions are dropped, and prove that it is the unique "plausible" mechanism that is in line with the recent and more precise mandates of the Supreme Court in *Saurav Yadav (2020)*.

School choice mechanisms using the celebrated individual-proposing deferred acceptance (DA) algorithm by Gale and Shapley (1962) as their core engines have been adopted by numerous jurisdictions worldwide in the last two decades following the formulation of this approach in Abdulkadiroğlu and Sönmez (2003). Under this approach, schools are endowed with exogenously specified choice rules that typically capture various objectives of the central planner. Following the same tradition, our proposed mechanism *2SMH-DA* is based on extending the 2SMH choice rule to the general version of problem with heterogenous positions across multiple institutions (or different departments of a single institution) through a joint implementation of the DA algorithm with the 2SMH choice rule for each institution. While our proposal is admittedly mainstream, its significance is in the constitutional basis we provide to 2SMH-DA by relating it to desiderata formulated in the Supreme Court's milestone ruling *Saurav Yadav (2020)*. As such, we argue that our paper not only provides a potential lead for the Supreme Court and the high courts in India with a precise instrument that can be used to remove the inconsistencies in the legal system which are partially documented in Section 5, but also provides the institutions in India with a mechanism to implement their reservation policies without any risk of generating controversial outcomes vulnerable for litigation.

Key for our advocacy for 2SMH-DA, *Saurav Yadav (2020)* brings clarity to two principles on joint implementation of VR and HR policies, elaborated in Section 2.5. While the judgment itself concerns a litigation for allocation of identical positions, the legal language that is used to describe these principles is suggestive that they are more broadly intended for the general version of the problem with heterogenous positions.[10] Postulating that the principles clarified in *Saurav Yadav (2020)* also reflect the Supreme Court's position more broadly for allocation of heterogenous positions, we argue that 2SMH-DA is the only "plausible" mechanism for the country. That is because, not only 2SMH-DA *Pareto dominates* any other mechanism that complies with the Saurav Yadav (2020) principles (Theorem 1), but it is also the only *strategy-proof* mechanism that complies with these principles (Theorem 2). Therefore, either one of the two fundamental principles in economic theory directly implies the 2SMH-DA mechanism when combined with *Saurav Yadav (2020)* formulation of the principles in *Indra Sawhney (1992)*. Hence we argue that 2SMH-DA is the only natural mechanism to address numerous legal challenges faced by public institutions in India due to their flawed allocation mechanisms.[11]

[10]As an illustrative example, see paragraph 31 in *Saurav Yadav (2020)*, given in Appendix C.1.

[11]While our analysis is motivated by India's legal and implementation challenges for its reservation system, our analytical results have policy relevance for applications in other countries as well. We are unaware

1.1. Root Cause of the Failures: A System Based on Migrations and Adjustments. As a secondary contribution of our paper, we identify the root cause of the inconsistencies between major court rulings on implementation of reservation policies in Section 5. As we have already emphasized, the VR policy was developed as the primary affirmative action policy to correct years of discrimination and to give a boost to groups historically discriminated against on the basis of caste in India. Since no individual can belong to more than a single caste (i.e., the protected groups cannot overlap), stand-alone implementation of the VR policy is straightforward with the *over-and-above choice rule* for the basic version of the problem with identical positions: First open positions are allocated based on merit, and then VR-protected positions of each reserved category are allocated to remaining members of the category based on *inter se merit* (i.e., merit within category members). Under this benchmark procedure, positions at each category is processed one-at-a-time, and starting with the open category.

In India, the legal language does not differentiate between categories for individuals and categories for positions. While individuals who do not belong to any VR-protected category are mainly referred to as *general-category* candidates, they are often referred to as *open-category* candidates as well. We find this terminology to be highly misleading and believe that it likely contributes to the confusion in India, because, unlike the positions in VR-protected categories which are exclusively set aside for the members of these categories, open-category positions are not exclusively set aside for the general-category candidates. Likely as a technical tool to avoid potential confusion in this and related issues, the concept of a *migration* has emerged in India. When a VR-protected individual receives an open-category position based on merit, she is considered to have *migrated* to the open or general category. In this way, just as positions set aside for the VR-protected categories are awarded to members of these categories, open-category positions also become awarded to members of the open category.

When the HR policy was later included in the system as a secondary reservation policy, in *Anil Kumar Gupta (1995)*, the Supreme Court responded by introducing a second technical concept of an *adjustment*. The idea is based on first determining a tentative assignment of positions at each category via the traditional over-and-above choice rule, and

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of any institution in other countries which implements the VR and HR policies concurrently. However, either one of these policies are implemented on a stand-alone basis in several applications worldwide. For example for the version of the problem with heterogenous positions, the VR policy is implemented for allocation of seats at Chicago's elite high schools (Dur et al., 2020) and the HR policy with overlapping protected groups is implemented in all cities of Chile for allocation of K-12 public school seats (Correa et al., 2019). For the version of problem with identical positions and overlapping protected groups, the Jordanian House of Representatives use a reservation system with 15 of the 130 seats reserved for women and 12 reserved for minorities (see `https://data.ipu.org/content/jordan?chamber_id=13434`, retrieved on 12/31/2021) and the National Assembly of Pakistan use a reservation system with 60 of the 342 seats reserved for women and 10 reserved for minorities (see `https://na.gov.pk/en/content.php?id=2`, retrieved on 12/31/2021).

subsequently making any necessary adjustments within each category to accommodate the minimum guarantees provided with the HR policy. However, until recently the adjustments for the open category positions were denied for the VR-protected individuals, thus resulting in the crisis documented in Sönmez and Yenmez (2021). This failure is recently corrected in *Saurav Yadav (2020)*, where the Supreme Court justices have clarified that VR-protected individuals are entitled for adjustments for open-category positions (just as they are eligible for them in the absence of HR protections).[12]

The tradition of relying on the concepts of migration and adjustment also persisted for the more general version of the problem with heterogenous positions. In a typical mechanism in the field, open positions across all institutions are tentatively allocated in a first phase through a *serial dictatorship* which assigns the highest merit individual her top choice, the second highest merit individual her top choice among the remaining open positions, and so on. Next, the VR-protected positions are tentatively allocated to eligible individuals in a similar way in a second phase. Since a VR-protected individual can receive two distinct tentative assignments in each of the first two phases, she migrates to the institution-category pair associated with her more preferred position. Adjustments for HR protections may be made at any stage. Subsequently, vacated positions have to be reallocated in a way that still respects merit for open positions and inter se merit for VR-protected positions. Key for our purposes, this equity objective is no longer feasible solely relying on the concepts of migration and adjustment, unless the underlying process indirectly captures them through an iterative procedure such as the DA algorithm.[13] For the simpler version of the problem with identical positions, relying on the concepts of migration and adjustment is not necessary to implement the 2SMH choice rule. That being the case, this choice rule does have an alternative (albeit more complicated) formulation which utilizes these concepts, and, therefore, it is feasible to use them in this framework.[14] However, for the general version of the problem with heterogeneous positions, relying on these concepts to derive an outcome precludes ones that reflect the principles in *Indra Sawhney (1992)*, unless an arbitrary number of rounds of migrations and adjustments are allowed. In our view, this methodological failure is at the core of the legal inconsistencies and ongoing litigations in India.[15] In Section 5, we show that the Supreme Court's main rulings for this version

[12]See paragraphs 24, 25, 31 and 32 in *Saurav Yadav (2020)*, given in Appendix C.1.

[13]The reason is analogous to the necessity of iterative procedures to find stable matchings in two-sided matching markets.

[14]In particular the formulation of the 2SMH choice rule endorsed in *Saurav Yadav (2020)* relies on the concepts of migration and adjustment. In contrast, Sönmez and Yenmez (2021) use a simpler formulation which does not rely on these concepts, and establishes its equivalence to the *Saurav Yadav (2020)* formulation in an Online Appendix.

[15]An especially detrimental aspect of this flawed methodology pertains to the reallocation of the open positions which are tentatively allocated to VR-protected individuals in the first phase earning them each a status called *meritorious reserved candidate*, but subsequently vacated when they migrate to a VR-protected

of the problem suffer from fundamental inconsistencies, both internally and also between themselves.

1.2. Related Literature. To the best of our knowledge, our paper is the first one to propose a mechanism to implement VR and HR policies in India concurrently for applications with heterogeneous positions (either at a single institution or across multiple institutions). As we have elaborated earlier in depth, our analysis builds on Sönmez and Yenmez (2021) where a simpler version of the problem with identical positions is considered. Apart from this paper, our motivation is closest to Ehlers and Morrill (2020) where the analysis is also based on legal desiderata, although our formulation is more direct in its reliance on explicit mandates in the judgments *Indra Sawhney (1992)* and *Saurav Yadav (2020)*. Our theoretical results generalize characterizations in Sönmez and Yenmez (2021) derived for identical positions only, and the basic characterizations in Alcalde and Barberà (1994), Balinski and Sönmez (1999) derived in the absence of any form of reservation policy.

Our paper is not the first one to suggest a mechanism based on the DA algorithm for practical applications of job matching or college admissions in India. Thakur (2018) adopts a similar approach for allocation of government positions by the Union Public Service Commission, whereas Baswana et al. (2019a) and Aygün and Turhan (2021) adopt similar approaches for allocation of seats at engineering colleges. However, unlike our paper, none of these papers properly accounts for the HR policy. Since HR protections for persons with disabilities is mandated throughout India by the full-bench Supreme Court judgement *Union Of India vs National Federation Of The Blind (2013)*,[16] inclusion of these provisions is essential for implementation of reservation policies. Two of these papers, Thakur (2018) and Aygün and Turhan (2021), assume away the HR policy altogether. In Baswana et al. (2019a), the authors report their design and implementation of a large scale seat allocation process for some of the technical universities in India.[17] While the authors incorporated both VR-protected and HR-protected groups in their design, they have not differentiated between the two policies under their mechanism. As such, their design is in direct violation of *Indra Sawhney (1992)*.[18]

category to receive a more-preferred position. A simple search via Indian Kanoon, a free search engine for Indian Law, reveals that as of January 2022, there were 28 cases at the Supreme Court and 670 more at the high courts which relate to the migration of meritorious reserved candidates.

[16]The ruling is available at `https://indiankanoon.org/doc/178530295/` (retrieved on 12/30/2021).

[17]Implemented in the period 2015-18, their system allocated approximately 39,000 seats at around 100 institutes in 2018 (Baswana et al., 2019b).

[18]HR protections are implemented as if they are VR protections under the mechanism designed and implemented by Baswana et al. (2019a,b). In particular, the specification of their mechanism treats persons with disabilities (PwD) as if they are a VR-protected group, even though they are explicitly awarded with HR protections by the Supreme Court judgement *National Federation Of The Blind (2013)*. For example, a general-category candidate with a disability is first considered for a regular open-category position, and only then for a PwD position at open category (see Figure 4.1 in Baswana et al. (2019b)). This treatment implies that a

Abstracting away from the legislative requirements in India, there is a large literature on priority-based resource allocation mechanisms using the DA algorithm as their core engine. A few papers in this literature are especially related to our paper since they include various forms of reservation policies. These include Dur et al. (2020) where the reservation policies are in the form of stand-alone VR protections, and Hafalir et al. (2013), Echenique and Yenmez (2015), Dur et al. (2018), and Abdulkadiroğlu and Grigoryan (2021) where the reservations are in the form of stand-alone HR protections. Other papers that study affirmative action for priority-based allocation mechanisms include Abdulkadiroğlu (2005), Haeringer and Klijn (2009), Calsamiglia et al. (2010), Kojima (2012), Westkamp (2013), Ehlers et al. (2014), Bó (2016), Doğan (2016), Kominers and Sönmez (2016), Fragiadakis and Troyan (2017), Combe (2018), and Hafalir et al. (2018).

2. Model

There exist a finite set of individuals $\mathcal{I}$ and a finite set of institutions $\mathcal{J}$ referred to as "jobs" throughout this section.[19] Each job $j \in \mathcal{J}$ has q_j identical positions. Each individual $i \in \mathcal{I}$ has a strict preference ranking $\succ_i$ over all jobs and the outside option denoted by $\emptyset$, which could be being unemployed. We denote the set of all preference rankings for agent i by $\mathcal{P}_i$. A job $j \in \mathcal{J}$ is *acceptable* to individual $i \in \mathcal{I}$ if it is stricty more preferred to the outside option, that is, $j \succ_i \emptyset$. We denote the corresponding weak order by $\succeq_i$ and the indifference relation by $\sim_i$. For any set of individuals $I \subseteq \mathcal{I}$, we denote the profile of individual preferences by $\succ_I = (\succ_i)_{i \in I}$. In addition, we denote the set of all preference profiles by $\mathcal{P} = (\mathcal{P}_i)_{i \in \mathcal{I}}$.

Each individual $i \in \mathcal{I}$ has a distinct merit score $\sigma_j(i) \in \mathbb{R}_+$ for any given job $j \in \mathcal{J}$. While individuals with higher merit scores have higher claims for a job in the absence of affirmative action policies, disadvantaged populations are protected with two types of affirmative action policies, (i) the *vertical reservation (VR)* policy providing the primary *VR protections*, and (ii) the *horizontal reservation (HR)* policy providing the secondary *HR protections*.

2.1. VR Policy. There exist a set of *VR-protected categories* $\mathcal{R}$ and a *general category* $g \notin \mathcal{R}$.[20] Each individual belongs to a single category in $\mathcal{R} \cup \{g\}$. Individual memberships to VR-protected categories is given by a function $\rho : \mathcal{I} \to \mathcal{R} \cup \{\emptyset\}$. Here, $\rho(i) = \emptyset$ indicates

position received by a disabled individual does not count against the reserved positions for PwD, contrary to the fundamental mandate on implementation of HR protections in *Indra Sawhney (1992)*.

[19]Our model and notation build on Sönmez and Yenmez (2021) where there is a single job with multiple identical positions.

[20]VR-protected categories are referred to as *reserved categories* in India.

that individual i is ineligible for VR protections, and thus she is a member of the general category g.[21]

At any given job $j \in \mathcal{J}$, there are $r_j^c \geq 0$ positions set aside exclusively for the members of category $c \in \mathcal{R}$. We refer to these positions as *category-c positions* or (*VR-protected positions for category c*). We assume that $\sum_{c\in\mathcal{R}} r_j^c \leq q_j$.[22] In contrast, members of the general category do not have any positions set aside for them under the VR policy. Therefore, $r_j^o = q_j - \sum_{c\in\mathcal{R}} r_j^c$ positions are open for all individuals. We refer to these positions as *open-category positions* (or *category-o positions*). Let $\mathcal{V} = \mathcal{R} \cup \{o\}$ denote the set of *vertical categories for positions*.

Definition 1. Given a VR-protected category $c \in \mathcal{R}$, an individual $i \in \mathcal{I}$ is *eligible for category-c positions* if,

$$\rho(i) = c.$$

Any individual $i \in \mathcal{I}$ is *eligible for open-category positions*.

Given a category $v \in \mathcal{V}$, let $\mathcal{I}^v \subseteq \mathcal{I}$ denote the set of individuals who are eligible for category-v positions.

The defining characteristic of the VR protections is stated as follows in the landmark Supreme Court judgement *Indra Sawhney (1992)*:

> It may well happen that some members belonging to, say Scheduled Castes get selected in the open competition field on the basis of their own merit; they will not be counted against the quota reserved for Scheduled Castes; they will be treated as open competition candidates.

When there is a single job and the VR policy is the only affirmative action policy, the interpretation of this statement becomes airtight: If a VR-protected individual deserves an open position on the basis of her merit score only, she should be awarded an open position and not use up a VR-protected position set aside for her category. In this sense, VR protections are implemented on an "over-and-above" basis, a feature which makes this policy the "higher level" affirmative action policy. Unfortunately, *Indra Sawhney (1992)* formulation of the VR policy given above loses its clarity when it is implemented jointly with the HR policy which is formally introduced next in Section 2.2.

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[21]To keep the notation at a minimum, we assume that (i) the set of VR-protected categories $\mathcal{R}$, (ii) the general category g, and (iii) the category-membership function ρ are all independent of a job. This assumption is without any loss of generality and all these primitives can be made job-dependent by a simple inclusion of a job index without interfering with any aspect of our analysis.

[22]In India, the total number of VR-protected positions cannot be more than half of the positions at any given job by the Supreme Court judgement *Indra Sawhney (1992)*, although this upper bound is not followed in some states such as Tamil Nadu.

2.2. HR Policy. In addition to the categories in $\mathcal{R}$ that are associated with the higher level VR protections, there is a finite set of traits $\mathcal{T}$ associated with the lower level HR protections. Each individual has a (possibly empty) subset of traits, given by the trait function $\tau : \mathcal{I} \to 2^{\mathcal{T}}$.

HR protections are provided in the form of minimum guarantees within each vertical category $v \in \mathcal{V}$.[23] For any job $j \in \mathcal{J}$, VR-protected category $c \in \mathcal{R}$, and trait $t \in \mathcal{T}$, subject to the availability of qualified individuals, a minimum of $r_j^{c,t}$ category-c positions are to be assigned to individuals from category c with trait t. If there are not enough individuals from category c with trait t to fill these positions, then the remaining empty seats are to be allocated to other individuals from category c. We refer to these positions as *category-c HR-protected positions for trait t*. Similarly, for any trait $t \in \mathcal{T}$ and subject to the availability of individuals with trait t, a minimum of $r_j^{o,t}$ open-category positions are to be assigned to individuals with trait t. If there are not enough individuals with trait t to fill these positions, then the remaining empty seats are to be allocated to other individuals. We refer to these positions as *open-category HR-protected positions for trait t*.

For each job $j \in \mathcal{J}$ and vertical category $v \in \mathcal{V}$, we assume that the total number of category-v HR-protected positions is no more than the number of positions in category v. That is, for each job $j \in \mathcal{J}$ and category $v \in \mathcal{V}$,

$$\sum_{t \in \mathcal{T}} r_j^{v,t} \leq r_j^v.$$

In contrast to VR protections which are provided on an "over-and-above" basis, HR protections are provided within each vertical category on a "minimum guarantee" basis. This means that positions obtained without invoking any HR protection still accommodate the HR protections.

2.3. Primary Assignment of Individuals to Jobs and Vertical Categories. In India, each position is classified by its job, vertical category (including the open category), and the associated trait (or its absence). Therefore, in order to describe an outcome, it may be compelling to assign individuals to a triple consisting of a job, a vertical category, and a trait or its absence. We will, however, take a slightly different approach for the reasons we elaborate below. An outcome needs to indicate the job assignments of individuals because they have strict preferences over jobs. While the category assignment is not important for individual preferences, it is important to implement the VR policy. That is because, the laws clearly specify who should receive the open positions and who should receive the

[23]Provision of HR protections within each vertical category is not a federal mandate in India but rather a formal recommendation by the Supreme Court judgment *Anil Kumar Gupta (1995)*. The vast majority of the institutions in India follows this recommendation and implement the HR policy in this form, which is also referred to as *interlocking reservations* or *compartmentalized horizontal reservations*.

VR-protected positions. Therefore, at a minimum, an outcome needs to specify the job assignment and the category assignment of each individual who receives a position. Specification of a trait assignment (or its absence), on the other hand, offers some flexibility in terms of modeling an outcome. While an outcome can explicitly specify the trait (or its absence) for a position that is received, this modeling choice results in immaterial multiplicities under our axioms which formulate affirmative action legislation in India. For example, if there is a minimum guarantee of two positions for women in the open category of a given job while there are five women who receive open-category positions, under Indian laws there is no meaningful way to specify which two of these five women receive the HR-protected positions. Therefore, in order to avoid any arbitrary conditions that fail to capture the Indian legislation, in our model an outcome simply assigns individuals to job-vertical category pairs or leaves them unassigned. An implicit trait assignment will still be important to verify that the HR protections are honored to the extent it is possible, and it will be captured in our model through a secondary assignment introduced in Section 2.4.[24]

Definition 2. An *assignment* is a function $\alpha : \mathcal{I} \to (\mathcal{J} \times \mathcal{V}) \cup \{\emptyset\}$ such that, for each $(j,v) \in \mathcal{J} \times \mathcal{V}$ and $I \subseteq \mathcal{I}$,

$$\alpha^{-1}(j,v) \subseteq I \cap \mathcal{I}^v \text{ and } |\alpha^{-1}(j,v)| \leq r_j^v.$$

We denote the set of all assignments by $\mathcal{A}$.

For each individual, an assignment specifies which job offers the position she receives, if any, and the vertical category through which she receives it. Given an assignment α, let

$$\alpha^{-1}(j) = \bigcup_{v \in \mathcal{V}} \alpha^{-1}(j,v)$$

denote the set of individuals who receive a position at job j.

Since individual preferences are originally defined over $\mathcal{J} \cup \{\emptyset\}$ rather than over $\big(\mathcal{J} \times \mathcal{V}\big) \cup \{\emptyset\}$, we trivially extend the domain $\mathcal{P}$ of the preferences to $\big(\mathcal{J} \times \mathcal{V}\big) \cup \mathcal{J} \cup \{\emptyset\}$ as follows: For any preference profile $\succ_{\mathcal{I}} \in \mathcal{P}$, individual $i \in \mathcal{I}$, job $j \in \mathcal{J}$, and category $v \in \mathcal{V}$, we have

$$j \succeq_i (j,v) \text{ and } (j,v) \succeq_i j,$$

or equivalently

$$(j,v) \sim_i j.$$

Therefore, an individual has preferences over jobs only and is indifferent between categories of the same job.

Definition 3. A *mechanism* $\varphi : \mathcal{P} \to \mathcal{A}$ is a function that selects an assignment $\varphi(\succ_{\mathcal{I}}) \in \mathcal{A}$ for each preference profile $\succ_{\mathcal{I}} \in \mathcal{P}$.

[24] This modeling choice allows us to relegate any immaterial multiplicities to a secondary assignment within the primary assignment.

Given a mechanism φ and and profile of reported preferences $\succ_{\mathcal{I}}$, the assignment for individual $i \in \mathcal{I}$ is denoted by $\varphi(\succ_{\mathcal{I}})(i)$. Likewise, the set of individuals assigned to job $j \in \mathcal{J}$ and category $v \in \mathcal{V}$ is denoted by $\varphi^{-1}(\succ_{\mathcal{I}})(j,v)$, and the set of individuals assigned to job j is denoted by $\varphi^{-1}(\succ_{\mathcal{I}})(j)$.

2.4. Secondary Assignment of Individuals to Traits within their Primary Assignments. In this section, we briefly discuss a technical tool we refer to as a *trait-matching* (Sönmez and Yenmez, 2021). Given an assignment $\alpha \in \mathcal{A}$, a trait-matching can be thought as a secondary assignment of the HR-protected individuals to traits within any given pair $(j,v) \in \mathcal{J} \times \mathcal{V}$, and its size provides us with a natural metric to assess to what extent the HR protections are honored at pair (j,v).

Fix an assignment $\alpha \in \mathcal{A}$, a job $j \in \mathcal{J}$, and a category $v \in \mathcal{V}$. Consider the set of individuals $I \subseteq \mathcal{I}$ who receive category-v positions at job j under assignment α. That is,

$$I = \alpha^{-1}(j,v).$$

First, consider a simpler version of the problem where each individual has at most one trait (i.e., the HR-protected groups do not overlap). For any trait $t \in \mathcal{T}$, the set of individuals in I who have trait t is given by $\{i \in I : t \in \tau(i)\}$. Therefore, within category v of job j,

- HR protections for trait t are fully honored if $|\{i \in I : t \in \tau(i)\}| \geq r_j^{v,t}$, whereas
- $r_j^{v,t} - |\{i \in I : t \in \tau(i)\}|$ of the trait-t HR-protected positions are left dishonored, otherwise.

For the latter case, an individual $i \in \mathcal{I}^v \setminus I$ can object to the allocation of category-v positions at job j under assignment α, provided that she has trait t and desires to receive a position at job j. Also observe that the total number of HR-protected positions that are honored within category v at job j by the set of individuals I is given by

$$n_j^v(I) = \sum_{t \in \mathcal{T}} \min\Big\{\big|\{i \in I : t \in \tau(i)\}\big|, r_j^{v,t}\Big\}.$$

Hence, any individual $i \in \mathcal{I}^v \setminus I$ can object to the allocation of category-v positions at job j under assignment α, provided that $n_j^v\big(I \cup \{i\}\big) > n_j^v(I)$ and she desires to receive a position at job j. This observation plays a key role in several of our formal axioms, later introduced in Section 2.5.

The same idea can also be extended to the more general version of the problem when individuals can have multiple traits (i.e., the HR-protected groups may potentially overlap). However, the secondary assignment of individuals to traits require additional care in this case. For example, if there is a single HR-protected position for women and a single HR-protected position for persons with disabilities, a disabled woman can receive positive discrimination for either one of the two HR-protected positions. However, if the only other individual who has either one of the two traits is a disabled man, it would be implausible to award the HR-protected position for persons with disabilities to the disabled woman and consequently deny an HR-protected position to the disabled man. Both of the HR-protected positions can be honored by awarding the HR-protected position for women to the disabled woman and the HR-protected position for persons with disabilities to the disabled man. We next build on this simple observation to extend the above-given function n_j^v to the general version of the problem.

Fix a job $j \in \mathcal{J}$, a category $v \in \mathcal{V}$, and a set of individuals $I \subseteq \mathcal{I}^v$. Let $H_j^{v,t}$ denote the set of HR-protected positions for trait-t within category v at job j, and $H_j^v = \bigcup_{t \in \mathcal{T}} H_j^{v,t}$ denote the set of all HR-protected positions within category v at job j.[25] Construct the following bipartite *HR graph*: Individuals in I are on one side of the graph and positions in H_j^v are on the other side. For any trait $t \in \mathcal{T}$, an individual $i \in I$ and a position $p \in H_j^{v,t}$ are *connected* in this graph if and only if individual i has trait t.

Definition 4. Given a job $j \in \mathcal{J}$, a category $v \in \mathcal{V}$, and a set of individuals $I \subseteq \mathcal{I}^v$, a *trait-matching* of individuals in I with HR-protected positions in H_j^v is a function $\mu : I \rightarrow H_j^v \cup \{\emptyset\}$ such that,

(1) for any $i \in I, t \in \mathcal{T}$,
$$\mu(i) \in H_j^{v,t} \implies t \in \tau(i),$$
(2) for any $i, j \in I$,
$$\mu(i) = \mu(j) \neq \emptyset \implies i = j.$$

Definition 5. Given a job $j \in \mathcal{J}$, a category $v \in \mathcal{V}$, and a set of individuals $I \subseteq \mathcal{I}^v$, a trait-matching μ of individuals in I with the HR-protected positions in H_j^v *has maximum cardinality in the HR graph* if there exists no other trait-matching that assigns a strictly higher number of HR-protected positions to individuals.

Let $n_j^v(I)$ denote the maximum number of job-j category-v HR-protected positions in H_j^v that can be assigned to individuals in I.[26] For any job $j \in \mathcal{J}$ and category $v \in \mathcal{V}$, this number identifies how many of its HR-protected positions are honored when the positions at category v of job j are awarded to individuals in I. As such, it serves as a key summary statistics on the compliance with the HR policy,[27] reflected in three of our formal axioms introduced next in Section 2.5.

[25]Note that, there are $r_j^{v,t}$ positions in the set $H_j^{v,t}$ for any trait $t \in \mathcal{T}$, and $\sum_{t \in \mathcal{T}} r_j^{v,t}$ positions in the set H_j^v.

[26]This number can be found in polynomial time by the famous *Hungarian maximum matching algorithm*, which is originally published in Kuhn (1955) and based on the earlier work of the Hungarian mathematicians Dénes König and Jenö Egerváry.

[27]This observation is the main reason why it it not necessary to explicitly include a trait matching in our modeling of an assignment.

2.5. Desiderata on Assignments and Mechanisms. In this section, we introduce our formal axioms on assignments and mechanisms.

Our main objective in formulation of these axioms is giving a mathematically precise meaning to the mandates of the Supreme Court of India on the reservation policy. We will organize our axioms into three groups.

(1) In the first group, we have two axioms which are so benign that they are not explicitly discussed in the court rulings.
(2) In the second group, we have three core axioms which formulate the Supreme Court's explicit mandates on concurrent implementation of VR and HR policies. As far as we can tell, the inability to design mechanisms which satisfy these axioms, and the legislative confusion on their formulation are the primary reasons for the challenges in India. We argue that the formulation of these three axioms in *Sauram Yadav (2020)* finally clears the second of these reasons; i.e., the legal confusion on their formulation. This landmark judgment of the Supreme Court also offered a mechanism for the simplest version of the problem with identical positions and non-overlapping HR-protected groups. In Section 3.3, we propose a mechanism for the problem in its full generality.
(3) When all positions are identical, the axioms in the first two groups uniquely characterize a mechanism (Sönmez and Yenmez, 2021). As we present in Example 2 below, this is not the case when positions are heterogenous. The role of the two axioms in the last group, namely the *Pareto principle* and *strategy-proofness*, is closing this modest gap. Together with the axioms that formulate the mandates of the Supreme Court, either one of these foundational principles in economic theory uniquely characterize our proposed mechanism *2SMH-DA* presented in Section 3.3.

Our first axiom states that no position should be awarded to an individual who has no desire to receive this position.

Definition 6. An assignment $\alpha \in \mathcal{A}$ satisfies *individual rationality* if, for every $i \in \mathcal{I}$,

$$\alpha(i) \succeq_i \emptyset.$$

A mechanism φ satisfies *individual rationality* if its outcome $\varphi(\succ_{\mathcal{I}})$ satisfies individual rationality for each $\succ_{\mathcal{I}} \in \mathcal{P}$.

Our second axiom states that, a position can be left idle only if no individual who desires to receive it is eligible for the position.

Definition 7. An assignment $\alpha \in \mathcal{A}$ satisfies *non-wastefulness* if, for every $j \in \mathcal{J}$, $v \in \mathcal{V}$, and $i \in \mathcal{I}$,

$$j \succ_i \alpha(i) \;\text{ and }\; \left|\alpha^{-1}(j,v)\right| < r_j^v \implies i \notin \mathcal{I}^v.$$

A mechanism φ satisfies *non-wastefulness* if its outcome $\varphi(\succ_{\mathcal{I}})$ satisfies non-wastefulness for each $\succ_{\mathcal{I}} \in \mathcal{P}$.

Our third axiom formulates the positive discrimination given to HR-protected individuals. It states that an individual cannot be denied a position at any job-category pair $(j, v) \in \mathcal{J} \times \mathcal{V}$, if her recruitment increases the number of HR-protected positions that are honored at pair (j, v).

Definition 8. An assignment $\alpha \in \mathcal{A}$ satisfies *maximal accommodation of HR protections* if, for every $j \in \mathcal{J}$, $v \in \mathcal{V}$, and $i \in \mathcal{I}^v$,

$$j \succ_i \alpha(i) \implies n_j^v\Big(\alpha^{-1}(j,v) \cup \{i\}\Big) \not> n_j^v\Big(\alpha^{-1}(j,v)\Big).$$

A mechanism φ satisfies *maximal accommodation of HR protections* if its outcome $\varphi(\succ_{\mathcal{I}})$ satisfies maximal accommodation of HR protections for each $\succ_{\mathcal{I}} \in \mathcal{P}$.

Our fourth axiom formulates the following equity principle given in the paragraph 31 of the Supreme Court judgment *Saurav Yadav (2020)*:

> `[...] Subject to any permissible reservations i.e. either Social (Vertical) or Special (Horizontal), opportunities to public employment and selection of candidates must purely be based on merit.`
>
> `Any selection which results in candidates getting selected against Open/General category with less merit than the other available candidates will certainly be opposed to principles of equality. There can be special dispensation when it comes to candidates being considered against seats or quota meant for reserved categories and in theory it is possible that a more meritorious candidate coming from Open/General category may not get selected. But the converse can never be true and will be opposed to the very basic principles which have all the while been accepted by this Court. Any view or process of interpretation which will lead to incongruity as highlighted earlier, must be rejected.`

Definition 9. An assignment $\alpha \in \mathcal{A}$ satisfies *no justified envy* if, for every $i \in \mathcal{I}$, $j \in \mathcal{J}$, $v \in \mathcal{V}$, and $i' \in \mathcal{I}^v$,

$$\left.\begin{array}{c}\alpha(i) = (j,v) \text{ and} \\ j \succ_{i'} \alpha(i')\end{array}\right\} \implies \Big\{ \sigma_j(i) > \sigma_j(i') \text{ or } n_j^v\Big(\alpha^{-1}(j,v)\Big) > n_j^v\Big(\big(\alpha^{-1}(j,v) \setminus \{i\}\big) \cup \{i'\}\Big).$$

A mechanism φ satisfies *no justified envy* if its outcome $\varphi(\succ_{\mathcal{I}})$ satisfies no justified envy for each $\succ_{\mathcal{I}} \in \mathcal{P}$.

That is, if an individual i receives a category-v position at job j while another individual i' receives a less-desired assignment, it is either because individual i has a higher merit score under σ_j than individual i' or because replacing individual i with individual i' decreases the number of HR-protected positions that are honored.

When applied for the open-category positions where every individual is eligible, our *no justified envy* axiom morphs into a condition that is known as the *principle of merit* in India. Similarly, when applied for the positions of a VR-protected category $c \in \mathcal{R}$ where only category-c individuals are eligible, our no justified envy axiom morphs into a condition that is known as the *principle of inter se merit* in India. While these principles were originally laid out in the landmark Supreme Court judgment *Indra Sawhney (1992)*, the precise role of the HR protections for open positions under the *principle of merit* only became airtight with the recent judgment *Saurav Yadav (2020)*.[28] Failure of this axiom is one of the primary reasons for litigations In India on implementation of the VR and HR policies.

Our fifth axiom formulates the defining characteristic of VR protections as the "higher level" reservation policy. It states that, VR-protected positions shall not be awarded to individuals who deserve an open-category positions based on the principle of merit, and thus be left for VR-protected individuals who are truly in need of positive discrimination.[29]

Definition 10. An assignment $\alpha \in \mathcal{A}$ satisfies *compliance with VR protections* if, for every $j \in \mathcal{J}$, $c \in \mathcal{R}$, and $i \in \mathcal{I}^c$, the following three conditions hold whenever $\alpha(i) = (j, c)$:

(1) $\left|\alpha^{-1}(j,o)\right| = r_j^o$,

(2) for every $i' \in \mathcal{I}$ with $\alpha(i') = (j,o)$,

$$\sigma_j(i') > \sigma_j(i) \text{ or } n_j^o\left(\alpha^{-1}(j,o)\right) > n_j^o\left((\alpha^{-1}(j,o) \setminus \{i'\}) \cup \{i\}\right), \text{ and}$$

(3) $n_j^o\left(\alpha^{-1}(j,o) \cup \{i\}\right) \not> n_j^o\left(\alpha^{-1}(j,o)\right)$.

A mechanism φ satisfies *compliance with VR protections* if its outcome $\varphi(\succ_{\mathcal{I}})$ satisfies compliance with VR protections for each $\succ_{\mathcal{I}} \in \mathcal{P}$.

Observe that, if any one of the three conditions in Definition 10 fails, then that means individual i is deserving of an open-category position: If the first condition fails, then there is an idle position at open category. If the second condition fails, then an open-category position is awarded to a less deserving individual i' who neither has a higher merit score

[28]In particular, the role of individuals who are both HR and VR protected for the open positions were unclear prior to *Saurav Yadav (2020)*. See, for example, the following quote from the 01/25/2021 *The Leaflet* article "Supreme Court strikes down policy of excluding the reserved community from competing for general and open category."

> Until now, the specific question of whether female candidates belonging to any of the vertically reserved categories can be selected on "merit" against the vacancies horizontally reserved for general/open category was a res integra before the Supreme Court.

The story is available in `https://tinyurl.com/z6y7wwfn`, last accessed on 02/09/2022.

[29]Since the exact formulation of the *principle of merit* as it is intended in *Indra Sawhney (1992)* only became clear with *Saurav Yadav (2020)*, the defining characteristic of the VR protections (in the presence of HR protections) also became clear only with this recent judgment.

than individual i, nor contributes to the number of HR-protected positions that are honored within open category. Lastly, if the third condition fails, then individual i deserves an open-category position because assigning him one increases the number of HR-protected positions that are honored within open category. Therefore, in this case individual i deserves an HR-protected position at open category on the basis of merit.[30]

When all positions are identical, i.e., when there is a single job, there is a unique mechanism that satisfies the five axioms given above, all mandated in India under *Saurav Yadav (2020)* (Sönmez and Yenmez, 2021). As we show in the next example, this is not the case when positions are heterogenous.[31]

Example 1. There are two jobs x, y, with one position each at open category. The only position at job x is HR-protected for trait t_1, and the only position at job y is HR-protected for trait t_2. There are two individuals a, b with traits $\tau(a) = \{t_1\}$ and $\tau(b) = \{t_2\}$. Preferences of the individuals, and their merit rankings at jobs are given as follows.

$\succ_a$	$\succ_b$
y	x
x	y
$\emptyset$	$\emptyset$

σ_x	σ_y
a	a
b	b

Observe that, both of the following two assignments satisfy all five axioms:

$$\alpha = \begin{pmatrix} a & b \\ (y,o) & (x,o) \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} a & b \\ (x,o) & (y,o) \end{pmatrix}.$$

While both individuals receive their first choices under assignment α, they receive their second choices under assignment β. □

Example 2 reveals that, in the absence of additional considerations, the HR policy may be detrimental to the very groups it is supposed to help. In particular, if assignment β is chosen by a central planner (which may be motivated by maximizing the number of HR-protected positions that are honored), the positive discrimination given to individual a for the position at job x due to her trait t_1 not only ends up hurting individual b, but also individual a herself. Indeed, having the highest merit score for both jobs, individual a is not in any need of positive discrimination. Therefore, an excessive effort to honor the HR-protected positions without considering individual preferences can be detrimental to the very groups the HR policy is supposed to protect. Fortunately, there is another assignment, which not only satisfies all five axioms representing the Supreme Court's mandates, but

[30] The role of the third condition in formulation of VR protections is due to *Saurav Yadav (2020)*. See Sönmez and Yenmez (2021) for additional details.

[31] Below we present the simplest possible example which makes this point, not one that is realistic. Modifying the example to make it realistic is straightforward.

also is in better interests of all individuals despite not honoring any of the HR-protected positions.

This observation motivates the axioms in our third group. While these axioms do not correspond to any desiderata formulated by the Supreme Court, they are among the most fundamental principles in economic theory.

Definition 11. An assignment $\alpha \in \mathcal{A}$ *Pareto dominates* assignment $\beta \in \mathcal{A}$ if,

(1) $\alpha(i) \succeq_i \beta(i)$ for all $i \in \mathcal{I}$, and
(2) $\alpha(i) \succ_i \beta(i)$ for some $i \in \mathcal{I}$.

A mechanism φ *Pareto dominates* a mechanism ϕ if,

(1) the assignment $\varphi(\succ_{\mathcal{I}})$ either Pareto dominates or is equal to the assignment $\phi(\succ_{\mathcal{I}})$ for each $\succ_{\mathcal{I}} \in \mathcal{P}$, and
(2) the assignment $\varphi(\succ_{\mathcal{I}})$ Pareto dominates the assignment $\phi(\succ_{\mathcal{I}})$ for some $\succ_{\mathcal{I}} \in \mathcal{P}$.

Definition 12. An assignment $\alpha \in \mathcal{A}$ is *Pareto efficient* if, there is no other assignment $\beta \in \mathcal{A}$ such that

(1) $\beta(i) \succeq_i \alpha(i)$ for all $i \in \mathcal{I}$, and
(2) $\beta(i) \succ_i \alpha(i)$ for some $i \in \mathcal{I}$.

A mechanism φ is *Pareto efficient* if its outcome $\varphi(\succ_{\mathcal{I}})$ is Pareto efficient for each $\succ_{\mathcal{I}} \in \mathcal{P}$.

Our final axiom is a highly sought-after *incentive compatibility* condition, defined only for mechanisms (and not for assignments).

Definition 13. A mechanism $\varphi : \mathcal{P} \to \mathcal{A}$ is *strategy-proof* if, for each $\succ_{\mathcal{I}} \in \mathcal{P}$, individual $i \in \mathcal{I}$, and $\succ'_i \in \mathcal{P}_i$,

$$\varphi(\succ_{\mathcal{I}})(i) \ \succeq_i \ \varphi(\succ'_i, \succ_{\mathcal{I}\setminus\{i\}})(i).$$

Truthful preference revelation is always weakly more preferred than reporting any other preference ranking for every individual under a strategy-proof mechanism.

3. Proposed Mechanism: 2SMH-DA

In this section, we extend a mechanism proposed in Sönmez and Yenmez (2021) for the version of the problem with identical positions to the general version with heterogenous positions through the celebrated *individual-proposing deferred acceptance algorithm* by Gale and Shapley (1962).

3.1. Single-Job Solution Concepts.

We next formulate a number of single-job solution concepts. Assuming *individual rationality* and a mechanism only relies on preferences over acceptable positions, the set of applicants contains all the necessary information that is given in a preference profile when all positions are identical. That is because, individuals who prefer remaining unmatched over receiving a position can be ignored, and the remaining individuals all have the same preference relation. Therefore, the domain of single-job solution concepts can be given as $2^{\mathcal{I}}$ rather than $\mathcal{P}$.

Definition 14. Given a job $j \in \mathcal{J}$ and a category $v \in \mathcal{V}$, a *single-category choice rule* is a function $C_j^v : 2^{\mathcal{I}} \to 2^{\mathcal{I}^v}$ such that, for any $I \subseteq \mathcal{I}$,

$$C_j^v(I) \subseteq I \cap \mathcal{I}^v \quad \text{and} \quad |C_j^v(I)| \leq r_j^v.$$

A single-category choice rule is simply the inverse mapping of a single job and single-category mechanism.

Definition 15. Given a job $j \in \mathcal{J}$, a *multi-category choice rule* is a multidimensional function $\vec{C}_j = (C_j^v)_{v \in \mathcal{V}} : 2^{\mathcal{I}} \to \prod_{v \in \mathcal{V}} 2^{\mathcal{I}^v}$ such that, for any $I \subseteq \mathcal{I}$,

(1) for any category $v \in \mathcal{V}$,

$$C_j^v(I) \subseteq I \cap \mathcal{I}^v \quad \text{and} \quad |C_j^v(I)| \leq r_j^v,$$

(2) for any two distinct categories $v, v' \in \mathcal{V}$,

$$C_j^v(I) \cap C_j^{v'}(I) = \emptyset.$$

A multi-category choice rule is simply the inverse mapping of a single job mechanism.

Definition 16. For any multi-category choice rule $\vec{C}_j = (C_j^v)_{v \in \mathcal{V}}$, the resulting *aggregate choice rule* $\widehat{C}_j : 2^{\mathcal{I}} \to 2^{\mathcal{I}}$ is such that, for any $I \subseteq \mathcal{I}$,

$$\widehat{C}_j(I) = \bigcup_{v \in \mathcal{V}} C_j^v(I).$$

For any set of individuals, the aggregate choice rule yields the set of chosen individuals across all categories.

3.2. 2-Step Meritorious Horizontal Choice Rule. In Sönmez and Yenmez (2021), we introduced the following single-category choice rule. Consider a job $j \in \mathcal{J}$ and a category $v \in \mathcal{V}$. Let $I \subseteq \mathcal{I}^v$ be a set of individuals who are eligible for category v.

Meritorious Horizontal Choice Rule $C^v_{Ⓜ,j}$

Step 1.1 Assuming such an individual exists, let i_1 be the the highest merit-score individual (with respect to σ_j) in I who has a trait for an HR-protected position. Choose individual i_1 for an HR-protected position. Let $I_1 = \{i_1\}$, and proceed with Step 1.2. If no such individual exists, proceed to Step 2.

Step 1.k ($k \in \{2, \ldots, \sum_{t \in \mathcal{T}} r_j^{v,t}\}$) Assuming such an individual exists, let i_k be the the highest merit-score individual (with respect to σ_j) in $I \setminus I_{k-1}$ with

$$n_j^v(I_{k-1} \cup \{i_k\}) = n_j^v(I_{k-1}) + 1.$$

Choose individual i_k for an HR-protected position. Let $I_k = I_{k-1} \cup \{i_k\}$, and proceed with Step 1.(k+1). If no such individual exists, proceed to Step 2.

Step 2. For unfilled positions, choose unassigned individuals with highest merit scores (with respect to σ_j) until either all positions are filled or all individuals are selected.

The following multi-category choice rule uses the meritorious horizontal single-category choice rule multiple times; first, to allocate open-category positions, and next for each VR-protected category to allocate VR-protected positions.

2-Step Meritorious Horizontal (2SMH) Choice Rule $\vec{C}^{2s}_{Ⓜ,j} = (C^{2s,v}_{Ⓜ,j})_{v\in\mathcal{V}}$

For each set of individuals $I \subseteq \mathcal{I}$,

$$C^{2s,o}_{Ⓜ,j}(I) = C^{o}_{Ⓜ,j}(I), \text{ and}$$
$$C^{2s,c}_{Ⓜ,j}(I) = C^{c}_{Ⓜ,j}\Big(\big(I \setminus C^{o}_{Ⓜ,j}(I)\big) \cap \mathcal{I}^c\Big) \quad \text{for any } c \in \mathcal{R}.$$

3.3. 2SMH-DA Mechanism. We are ready to present our proposed mechanism, which extends the 2SMH mechanism–defined for identical positions–to the general version of the problem with heterogenous positions with the celebrated individual-proposing deferred acceptance algorithm (Gale and Shapley, 1962).

2SMH-DA Mechanism $\varphi^{2s}_{Ⓜ}$

For each preference profile $\succ_{\mathcal{I}} \in \mathcal{P}$, the outcome $\varphi^{2s}_{Ⓜ}(\succ_{\mathcal{I}})$ of the 2SMH-DA mechanism is obtained as follows:

Step 1. Assuming such a job exists, each individual $i \in \mathcal{I}$ applies to her most preferred acceptable job under $\succ_i$. Let I^1_j be the set of individuals who apply to job $j \in \mathcal{J}$. Each job $j \in \mathcal{J}$ tentatively assigns individuals in $\widehat{C}^{2s}_{Ⓜ,j}(I^1_j)$ to its categories based on $\vec{C}^{2s}_{Ⓜ,j}(I^1_j)$, and (permanently) rejects any remaining applicants. If there is no rejection by any job, then the procedure is terminated and the tentative assignments are finalized. Otherwise, proceed to Step 2.

Step k. Assuming such a job exists, each individual $i \in \mathcal{I}$ who is rejected in Step $(k-1)$ applies to her next preferred acceptable job under $\succ_i$. For any job $j \in \mathcal{J}$, let I^k_j be the set of new applicants in Step k along with individuals who are tentatively assigned to categories of job j in Step $(k-1)$. Each job $j \in \mathcal{J}$ tentatively assigns individuals in $\widehat{C}^{2s}_{Ⓜ,j}(I^k_j)$ to its categories based on $\vec{C}^{2s}_{Ⓜ,j}(I^k_j)$, and (permanently) rejects any remaining applicants. If there is no rejection by any job, then the procedure is terminated and the tentative assignments are finalized. Otherwise, proceed to Step (k+1).

Since there is a finite number of jobs and individuals, this mechanism terminates at a finite round for every preference profile.

4. Main Results

We next present our main results. The general message of our results is, while the five axioms formulated in *Saurav Yadav (2020)* do not single out a mechanism when positions are heterogenous, they come pretty close, and either one of two fundamental axioms in economic theory completes this gap.

Theorem 1. *2SMH-DA Pareto dominates any other mechanism that satisfies*

(1) individual rationality,
(2) non-wastefulness,
(3) maximal accommodation of HR protections,
(4) no justified envy, and
(5) compliance with VR protections.

The primary objective of the Indian reservation system is to enhance the social and educational status of underprivileged communities and thus improve their lives. Given Theorem 1 we believe 2SMH-DA is the only plausible mechanism to pursue that objective.

Observe that, Theorem 1 does not imply that the mechanism 2SMH-DA is *Pareto efficient*. Our next example shows that, Pareto efficiency is incompatible with the Supreme Court's axioms even when all institutions have the same merit ranking of individuals.[32]

Example 2. There are two jobs x, y, with one position each at open category. The only position at job x is HR-protected for trait t_1, and the only position at job y is HR-protected for trait t_2. There are three individuals a, b, c with traits $\tau(a) = \{t_1\}$ and $\tau(b) = \tau(c) = \{t_2\}$. Preferences of the individuals, and their merit rankings at jobs are given as follows.

$\succ_a$	$\succ_b$	$\succ_c$
y	x	x
x	y	y
$\emptyset$	$\emptyset$	$\emptyset$

σ_x	σ_y
a	a
b	b
c	c

Consider the following two assignments:

$$\alpha = \begin{pmatrix} a & b & c \\ (x,o) & (y,o) & \emptyset \end{pmatrix} \quad \text{and} \quad \beta = \begin{pmatrix} a & b & c \\ (y,o) & (x,o) & \emptyset \end{pmatrix}.$$

[32] It is well-known that, even in the absence of VR and HR protections, the axioms of *individual rationality*, *non-wastefulness*, and *no-justified envy* are incompatible with *Pareto efficiency* (Alcalde and Barberà, 1994; Balinski and Sönmez, 1999). However, *Pareto efficiency* is compatible with the axioms of *individual rationality*, *non-wastefulness*, and *no-justified envy* when merit ranking of individuals is identical across all institutions. (Balinski and Sönmez, 1999).

Since (i) all individuals prefer either of the two jobs to remaining unmatched, (ii) individuals b and c each have trait t_2, and (iii) the only position at job y is HR-protected for trait t_2, individual a cannot receive his first choice position at job y under any assignment that satisfies the axiom of maximal accommodation of HR protections. Therefore, assignment α is the only assignment that satisfies all five axioms, even though it is Pareto dominated by assignment β. □

While it is perhaps the most fundamental axiom in economic theory, the *Pareto principle* is not the only condition that closes the gap between the *Saurav Yadav (2020)* axioms and our proposed mechanism 2SMH-DA. Our final result shows that *strategy-proofness* also achieves the same task, further strengthening the case for 2SMH-DA.

Theorem 2. *A mechanism satisfies*

(1) individual rationality,
(2) non-wastefulness,
(3) maximal accommodation of HR protections,
(4) no justified envy,
(5) compliance with VR protections, and
(6) strategy-proofness

if, and only if, it is 2SMH-DA.

5. Legal Inconsistencies within and between Supreme Court Judgements

In this section we present the legal inconsistencies that emerged in several important judgments of the Supreme Court, and identify a methodological flaw that is largely responsible for these inconsistencies.[33] In all judgements presented in this section, the VR-protected categories are SC, ST, and OBC.

5.1. Meritorious Reserved Candidates. While the specific mechanism differs in most applications, the initial step of mechanisms employed by various institutions in India often consists of tentatively allocating the $\sum_{j \in J} r_j^o$ open-category positions to candidates using a mechanism known as the *serial dictatorship* in the literature: The highest merit ranking

[33]These inconsistencies often result in litigations, interruption of the recruitment processes, and reversals of recruitment decisions in India. For example, a March 2017 *The Times of India* story reports the likely consequences of a ruling by the High Court of Gujarat as follows:

> " The advertisement was issued in 2010 and recruitment took place in 2016 amid too many litigations over the issue of reservation ... With the recent observation by the HC, the merit list will now be changed for the third time. Those already selected and at present under training might lose their jobs, and half a dozen new candidates might find their names on the new list. However, all appointments have been made by the HC conditionally and subject to final outcome of these multiple litigations."

The story (retrieved on 01/06/2022) is available at `https://tinyurl.com/995xykym`.

candidate tentatively receives his top choice job, the second highest merit ranking candidate tentatively receives his top choice job among the remaining open positions, and so on. Each reserved category candidate who tentatively receives an open position at this step is referred to as a *meritorious reserved candidate* (MRC).

Consider an individual i who is an MRC from a VR-protected category, say from SC. Observe that, while candidate i tentatively receives an open-category position on his own merit without using the benefits of VR protections, this position is not necessarily at his first choice job. Therefore, he would rather receive an SC-category position at a more-preferred job. At this point, the following important questions emerge, where the answers guide the mechanics of the rest of the mechanism:

(1) Shall an MRC who is tentatively assigned an open-category position be allowed to *migrate* to a higher choice job, and receive a position set aside for his VR-protected category?
(2) If the answer to the first question is in the positive, then what is to happen to the open-category position that is vacated by the MRC?

These two questions and their answers are at the heart of countless lawsuits in India. We next present three Supreme Court cases in this context. Through these cases we argue that the methodology of using migrations and adjustments through meritorious reserved candidates is fundamentally flawed, and it is the main source of the legal conflict and confusion in all of these cases and countless others. All these difficulties can be avoided with our proposed mechanism 2SMH-DA, presented in Section 2.[34]

5.2. Anurag Patel vs U.P. Public Service Commission (2004). The Uttar Pradesh Public Service Commission (UPPSC) conducted an examination in 1990, merit ranking all candidates, and used the following mechanism to allocate 358 positions at various jobs:

Step 1. Allocate the $\sum_{j\in J} r^o_j$ units of open-category positions using the serial dictatorship induced by the given merit ranking: The highest merit ranking candidate receives his top choice, the second highest merit ranking candidate receives his top choice among the remaining open positions, and so on.

[34]Each Supreme Court case in Sections 5.2-5.4 involves the handling of MRC candidates under a litigated mechanism in India. The descriptions of the mechanisms we present in these sections are based on their descriptions in these court cases. Not all aspects of the actual mechanisms are relevant for these cases, and they only provide details that relate to the case. In particular, all the cases focus on VR protections to SC/ST/OBC and none of them gives details on the handling of HR protections as they are not focal to these cases. This means that the mechanisms we present may correspond to a simplified case, abstracting away from HR protections. In actual implementation HR protections are likely accommodated through adjustments at various steps of the procedures, as it is traditional in India. Since we present failures of these mechanisms in this section even in the absence of HR protections, the details provided in the cases are sufficient for our purposes. However, there are other cases where the litigation involves both HR protections and also heterogenous positions across multiple institutions. See, for example, the Patna High Court case *The Controller Of Exam.,Bihar vs Nidhi Sinha & Anr*, available at `https://indiankanoon.org/doc/180601564/` (retrieved on 01/04/2022).

All assignments in this step are final.

Step 2. For each VR-protected category $c \in \{SC, ST, OBC\}$, consider only category-c candidates who have not received an assignment in Step 1, and allocate the $\sum_{j \in J} r_j^c$ units of category-c positions to these candidates using the serial dictatorship induced by the given merit ranking.

All assignments in this step are final.

At least one of the shortcomings of this mechanism is immediately apparent: MRC candidates who receive their assignments in Step 1 are not given an opportunity to migrate and be considered for any of the VR-protected positions for their categories, and as such they often receive positions at less-preferred jobs compared to lower merit ranking candidates from their own categories. Therefore, the UPPSC mechanism fails to respect *inter se merit*, an important principle that plays a key role in all Supreme Court cases we discuss in Sections 5.2-5.4. This shortcoming of the UPPSC mechanism resulted in a lawsuit at the High Court of Allahabad, and consequently the UPPSC was ordered to come up with a reallocation that respects *inter se* merit. This reallocation, in turn, resulted in an appeal at the Supreme Court by a candidate who was adversely affected by the high court's decision. The appeal was dismissed by the Supreme Court, and the high court's decision was sustained, reaffirming that the mechanism has to respect *inter se* merit. The following quote is from this important judgement:

> In the instant case, as noticed earlier, out of 8 petitioners in writ petition No. 22753/93, two of them who had secured ranks 13 and 14 in the merit list, were appointed as Sales Tax Officer-ll whereas the persons who secured rank Nos. 38, 72 and 97, ranks lower to them, got appointment as Deputy Collectors and the Division Bench of the High Court held that it is a clear injustice to the persons who are more meritorious and directed that a list of all selected backward class candidates shall be prepared separately including those candidates selected in the general category and their appointments to the posts shall be made strictly in accordance with merit as per the select list and preference of a person higher in the select list will be seen first and appointment given accordingly, while preference of a person lower in the list will be seen only later.

Anurag Patel (2004) is best known for reaffirming that any mechanism used for allocation of government jobs or seats at public educational institutions has to respect *inter se* merit.[35] Therefore, an MRC is entitled by law to migrate to a higher choice job claiming a position

[35] *Anurag Patel (2004)* also supports our position that, the principles on implementation of VR and HR policies clarified in *Saurav Yadav (2020)* is not limited to applications with identical positions, but they apply more broadly for applications with heterogenous positions as well.

vertically reserved for his reserved category, answering the first question in Section 5.1 in the positive.

5.3. Union of India vs Ramesh Ram & Ors (2010). Selection to three *All India Services* (Indian Administrative Service, Indian Foreign Service, and Indian Police Service), and eighteen other services in various government departments is made by the Union Public Service Commission (UPSC), by conducting Civil Service Examinations periodically. Given the merit ranking produced by the Civil Service Examination along with the submitted preferences of the candidates over the set of jobs, the following *UPSC mechanism* is used to allocate the positions.

Step 1. Tentatively allocate the $\sum_{j \in J} r_j^o$ units of open-category positions using the serial dictatorship induced by the given merit ranking. Promote the VR-protected candidates who secured tentative positions at this step to the status of an MRC.

Finalize all tentative assignments, except those received by VR-protected candidates who are promoted to the status of MRC.

Step 2. For each VR-protected category $c \in \{SC, ST, OBC\}$, consider all category-c candidates (including MRCs who each received a tentative assignment in Step 1), and tentatively allocate the $\sum_{j \in J} r_j^c$ units of category-c positions to these candidates using the serial dictatorship induced by the given merit ranking.

Finalize all tentative assignments except those received by the MRCs.

Step 3. Let m^c denote the number of MRCs from the VR-protected category $c \in \{SC, ST, OBC\}$. Restricting attention to candidates who have not received an assignment (tentative or final) in Step 1 or Step 2, prepare the following four waitlists:

(1) General category waitlist: $(m^{SC} + m^{ST} + m^{OBC})$ highest merit ranking general category candidates.
(2) Category-SC waitlist: m^{SC} highest merit ranking candidates from SC.
(3) Category-ST waitlist: m^{ST} highest merit ranking candidates from ST.
(4) Category-OBC waitlist: m^{OBC} highest merit ranking candidates from OBC.

Step 4. Finalize the assignment of each MRC with the more-preferred one of the (at most) two tentative assignments received in Steps 1 and 2. In case the two tentative assignments correspond to the same job, finalize the open-category position received in Step 1.

Step 5. For each MRC, (at most) one position may be vacated at Step 4 and become available for reassignment. Allocate them to waitlisted candidates as follows:

(i) For each MRC whose assignment is finalized as the VR-protected position he received in Step 2, the open-category position he received in Step 1 becomes vacant. Allocate these vacated open-category positions to candidates in the general category waitlist with the serial dictatorship induced by the merit ranking.
(ii) For each MRC from category $c \in \{SC, ST, OBC\}$ whose assignment is finalized as the open-category position he received in Step 1, a category-c position may be vacated in case the MRC tentatively received one in Step 2. Allocate these vacated category-c positions to candidates in the category-c waitlist with the serial dictatorship induced by the merit ranking.

UPSC declares the results in two stages: Steps 1-3 in first stage, and Steps 4, 5 in the second stage. Under their mechanism, the MRC-related questions posed in Section 5.1 are handled as follows:

(1) An MRC is allowed to migrate to a preferred job, claiming a VR-protected position for his category.
(2) An open-category position tentatively assigned to an MRC in Step 1 is awarded to a waitlisted candidate from the general category once the MRC receives a more-preferred position that is VR protected.

The legality of the UPSC mechanism was scrutinized at each of the three levels of the Indian Judicial System. First, a number of OBC candidates (each of whom failed to receive an assignment despite being waitlisted) filed several applications at various branches of the Central Administrative Tribunal, challenging the UPSC mechanism. They argued that MRCs shall not be allowed to migrate to a higher choice job, claiming positions vertically reserved for SC/ST/OBC candidates. Their position is articulated in a later Supreme Court judgement *Ramesh Ram (2010)* as follows:[36]

> `It was contended that adjustment of OBC merit candidates against OBC reservation vacancies was illegal. According to them, such candidates should be adjusted against the general (unreserved) vacancies, as that would have allowed more posts for OBC candidates and would have allowed the lower ranked OBC candidates a better choice of service. They contended that more meritorious OBC candidates should be satisfied with lower choice of service as they became general (unreserved) candidates by reason of their better performance.`

Of course, the petitioners' position is against the principle of *inter se* merit and in direct conflict with the Supreme Court judgement in *Anurag Patel (2004)* discussed in Section 5.2. Despite the unsustainable position taken by the petitioners, their case was not dismissed by

[36]See the June 2010 *Frontline* story "Bringing Clarity," available at `https://frontline.thehindu.com/static/html/fl2712/stories/20100618271210300.htm` (retrieved on 06/01/2022).

the Tribunal. The Tribunal instead ruled that, while the MRCs can be allowed to migrate to a higher choice job claiming positions that are vertically reserved for their categories, this shall not be done at the expense of consuming away the VR-protected positions for categories SC, ST, and OBC. In other words, while the petitioners' challenged Step 1 of the UPSC mechanism, the Tribunal required the UPSC to change Steps 2, 3, and 5 of its mechanism.

This ruling was challenged by the Union of India at the Madras High Court. Not only did the Union of India lose their appeal in a judgement upholding the Tribunal's decision, the High Court ruled the following aspect of the UPSC mechanism to be unconstitutional:

```
Rule 16.(2):  While making service allocation, the candidates belonging
to the Scheduled Castes, the Scheduled Tribes or Other Backward Classes
recommended against unreserved vacancies may be adjusted against reserved
vacancies by the Govt.  if by this process they get a service of higher
choice in the order of their preference.
```

This corresponds to ruling Steps 2, 3, and 5 of the UPSC mechanism to be unconstitutional. Consequently, the High Court directed the Government of India and UPSC to repeat the allocation in the absence of their Rule 16(2).

The judgement of the Madras High Court, in turn, was challenged by the Union of India at the Supreme Court in *Ramesh Ram (2010)*. In a decree that became a main reference for the allocation of government positions, the appeal was allowed, the judgement of the Madras High Court was set aside, and the UPSC mechanism was ruled to be constitutional. The following statement is from the conclusion of this historical decree:

```
We sum up our answers-:
  i) MRC candidates who avail the benefit of Rule 16 (2) and adjusted in
the reserved category should be counted as part of the reserved pool for the
purpose of computing the aggregate reservation quotas.  The seats vacated
by MRC candidates in the General Pool will be offered to general category
candidates.
  ii) By operation of Rule 16 (2), the reserved status of an MRC is
protected so that his/ her better performance does not deny him of the
chance to be allotted to a more-preferred service.
  iii) The amended Rule 16 (2) only seeks to recognize the inter se merit
between two classes of candidates i.e.  a) meritorious reserved category
candidates b) relatively lower ranked reserved category candidates, for the
purpose of allocation to the various Civil Services with due regard for the
preferences indicated by them.
  iv) The reserved category candidates ''belonging to OBC, SC/ ST
categories'' who are selected on merit and placed in the list of
General/Unreserved category candidates can choose to migrate to the
```

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respective reserved category at the time of allocation of services.  Such
migration as envisaged by Rule 16 (2) is not inconsistent with Rule 16 (1)
or Articles 14, 16 (4) and 335 of the Constitution.
```

Therefore, in the context of allocation of government jobs, the Supreme Court judgement *Ramesh Ram (2010)* provides the following answers to the questions posed in Section 5.1:

(1) An MRC is entitled to migrate to a higher choice job claiming a VR-protected position for his category.
(2) The open-category positions vacated by MRCs are to be offered to the general category candidates.

The judges of the Supreme Court justified this important decision based on the principle of *inter se* merit, reaffirming the judgement in *Anurag Patel (2004)*. However, there is an important oversight in their judgement, one which makes the UPSC mechanism unconstitutional. While the Supreme Court overruled the judgement by the Madras High Court, justifying their decision based on the principle of *inter se* merit, the judges of the Supreme Court failed to observe that the UPSC mechanism itself does not comply with this important principle. The following simple example makes this point.

Example 3. There are three jobs x, y, z. Each job has one open-category position and job x has an additional VR-protected position for category c. There are five candidates a_1, a_2, a_3, b_1, b_2. Candidates b_1, b_2 are members of category-c and hence they are eligible for the VR-protected position. Candidates a_1, a_2, a_3 are members of the general category, and therefore ineligible for the VR-protected position. All candidates have the same preferences where x is their first choice, y is their second choice, z is their third choice, and remaining unmatched is their last choice.

All jobs have the same merit ranking of the candidates based on the merit function σ as follows:

$$\sigma(a_1) > \sigma(b_1) > \sigma(a_2) > \sigma(b_2) > \sigma(a_3).$$

We next find the outcome of the UPSC mechanism:

Step 1. The highest merit ranking candidate a_1 tentatively receives an open position at job x, the second highest merit ranking candidate b_1 receives an open position at job y, and the third highest merit ranking candidate a_2 receives an open position at job z.

Candidate b_1 is given the status of an MRC. Assignment of candidate a_1 is finalized as an open position at job x, assignment of candidate a_2 is finalized as an open position at job z.

Step 2. Candidates b_1 and b_2 are the only ones eligible for the category-c position at job x. Having higher merit ranking than candidate b_2, candidate b_1 tentatively receives this position.

Step 3. A waitlist each is prepared for the general category and category-c. Since there is only one MRC candidate, there is a single candidate in each waitlist. Candidates a_3 and b_2 are waitlisted at the general category waitlist and category-c waitlist respectively.

Step 4. Having the status of an MRC, the assignment of candidate b_1 is finalized as the more-preferred position he tentatively received from Steps 1 and 2. He receives the category-c position at his first choice job x.

Step 5. The position vacated by candidate b_1 is an open-category position at job y. It is assigned to candidate a_3 as the only individual in the general category waitlist.

Therefore, the final assignment is given as follows:

$$\begin{pmatrix} a_1 & a_2 & a_3 & b_1 & b_2 \\ (x,o) & (z,o) & (y,o) & (x,c) & \emptyset \end{pmatrix}.$$

Observe that this assignment does not respect *inter se* merit. Candidate a_2 receives a less-preferred assignment than candidate a_3, despite being a member of the same category (i.e., the general category) and having a higher merit score. □

Indeed, a close inspection of Example 3 reveals a number of additional issues with the judgement in *Ramesh Ram (2010)*. The Supreme Court ruled that:

> The seats vacated by MRC candidates in the General Pool will be offered to general category candidates.

This decision would be plausible only if candidates from the general category are more meritorious than those in the VR-protected categories. As it is seen in Example 3, this may not always be the case. In our view, offering the vacated position to the lowest merit ranking candidate a_3 is not justified when the higher merit ranking candidate b_2 remains unassigned simply because he is a member of a VR-protected category. A system that is intended as positive discrimination for candidate b_2 results in his discrimination. Equivalently, the *cut-off score*, the minimum score needed for a position, is higher in this example for the category-c candidates than for the general category candidates.[37] These types of scenarios result in some other related anomalies as well. In the absence of affirmative action, the outcome of the UPSC mechanism would have been

$$\begin{pmatrix} a_1 & a_2 & a_3 & b_1 & b_2 \\ x & y & \emptyset & x & z \end{pmatrix},$$

and the sole VR-protected candidate b_2 would have been better off. Or, alternatively, had candidate b_2 not claimed his VR protections, he would have received a position at job z.

[37] October 2019 *ThePrint* story "Why civil services exams in some states have had higher cut-offs for SC/ST & OBC applicants" gives a real-life example of this failure. See `https://tinyurl.com/y9x4mbuw` (retrieved on 01/06/2022).

5.4. Tripurari Sharan & Anr. vs Ranjit Kumar Yadav (2018). The judgement in *Ramesh Ram (2010)*, discussed in Section 5.3, is now considered a main reference for allocation of government jobs when positions are heterogeneous. Based on this reference judgement, open-category seats vacated by MRC candidates are to be offered to general-category candidates for allocation of government jobs. We emphasize *government jobs*, because the Supreme Court has taken a contrary position for the allocation of seats at medical colleges. While the main reference for this application is considered to be *Shri Ritesh R. Sah vs Dr. Y.L. Yamul & Ors (1996)*, we instead discuss the more recent Supreme Court case *Tripurari Sharan (2018)*,[38] for it is more illuminating for our purposes.

Citing the judgement in *Ramesh Ram (2010)*, the petitioners appealed in *Tripurari Sharan (2018)* an earlier decision by the Patna High Court, which ruled:

> In case of admission to medical institutions, an MRC can have in, for the purpose of allotment of institutions, of his choice, the option of taking admission in a college, where a seat in his category is reserved. Though admitted against a reserved seat, for the purpose of computation of percentage of reservation, he will be deemed to have admitted as an open category candidate, rather he remains an MRC. He cannot be treated to have occupied a seat reserved for the category of reservation he belongs to. Resultantly, this movement will not lead to ouster of the reserved candidate at the bottom on the list of that reserved category. While his/her selection as reserved category candidate shall remain intact, he/she will have to adjusted against remaining seats, because of movement of an MRC against reserved seats, only for the purpose of allotment of seats.

Aware of the contradictory judgement in *Ramesh Ram (2010)*, the judges of the Patna High Court justified their decision as follows:

> (i) There is an obvious distinction between qualifying through a common entrance test for securing admission to medical courses in various institutions vis-a-vis a common competitive examination held for filling up vacancies in various services.
> (ii) This distinction arises because all candidates receive, in a case of common entrance test held for securing admission in medical institutions, the same benefits of securing admission in one of the medical institutions, in a particular course, whereas in the case common selection process adopted for filling up vacancies in various services, there are variations, which accrue to the successful candidates, because the services may differ in terms of status and conditions of service including pay scale, promotional avenues, etc. Consequence of migration of an MRC to the concerned reserved

[38] The ruling is available at `https://indiankanoon.org/doc/102870864/` (retrieved on 01/06/2022).

```
category shall be, therefore, different in case of the admission to various
medical institutions vis-a-vis selection to various posts.
```

According to the judges, while the benefits from securing different jobs may vary, the benefits from securing admission to different medical institutions are uniform. We do not agree with this assessment; however, even if that is the case, then why bother migrating an MRC to a higher choice medical institution?

The appeal was declined by the Supreme Court in *Tripurari Sharan (2018)*, reaffirming the Patna High Court's decision. Furthermore, the Supreme Court judgement also specified the exact manner in which the open-category seats vacated by MRC candidates are to be filled in allocation to medical institutions:

```
  i) An MRC can opt for a seat earmarked for the reserved category, so as to
not disadvantage him against less meritorious reserved category candidates.
Such MRC shall be treated as part of the general category only.
  ii) Due to the MRC's choice, one reserved category seat is occupied, and
one seat among the choices available to general category candidates remains
unoccupied.  Consequently, one lesser-ranked reserved category candidate who
had choices among the reserved category is affected as he does not get any
choice anymore.
  To remedy the situation i.e.  to provide the affected candidate a remedy,
the 50th seat [intended as the last reserved position] which would have
been allotted to X-MRC, had he not opted for a seat meant for the reserved
category to which he belongs, shall now be filled up by that candidate in
the reserved category list who stands to lose out by the choice of the MRC.
```

So an MRC is allowed to migrate to a VR-protected seat at a higher choice college in order to respect *inter se* merit, and the open-category seat vacated by the MRC is to be awarded to the VR-protected candidate who is displaced due to this migration. There are numerous issues with this judgement, including its contradiction with *Ramesh Ram (2010)*. But perhaps the most striking one is, the following inconsistency in the final judgement quoted above: While the judges justify part (i) above on the basis of *inter se* merit, they fail to observe that their mandate in part (ii) results in a potential compromise of *inter se* merit! As such, this judgement contradicts with *Anurag Patel (2004)* as well. This is the main point made in our next example.

Example 4. There are two colleges x and y. College x has two open-category seats and two VR-protected seats for category c. College y has one open-category seat only. There are five candidates a_1, a_2, b_1, b_2, b_3. Candidates b_1, b_2, b_3 are members of category-c and hence they are eligible for the VR-protected position. Candidates a_1, a_2 are members of the general category, and therefore ineligible for the VR-protected position.

Preferences of the candidates are are given as follows.

$\succ_{a_1}$	$\succ_{a_2}$	$\succ_{b_1}$	$\succ_{b_2}$	$\succ_{b_3}$
x	x	x	y	y
y	y	y	x	x

Both schools have the same merit ranking of the candidates, given by the merit function σ as follows:

$$\sigma(a_1) > \sigma(a_2) > \sigma(b_1) > \sigma(b_2) > \sigma(b_3).$$

While the mechanisms of various medical colleges may differ, they all produce the same assignment in this example, provided that they comply with the judgement in *Tripurari Sharan (2018)*. The three open-category seats are allocated to the highest merit score candidates, where the general category candidates a_1, a_2 each receive an open-category seat at college x, and the category-c candidate b_1 tentatively receives an open-category seat at college y. Receiving a seat on his own merit, category-c candidate b_1 is promoted to the status of an MRC. The two category-c seats at college x are tentatively allocated to the two remaining candidates b_2 and b_3 from category c. At this stage, the court decision in *Tripurari Sharan (2018)* kicks in. Candidate b_1 who is promoted to the status of an MRC prefers a seat at college x to his tentative assignment at college y. Therefore, he is assigned one of these seats at the expense of the lowest merit ranking category-c candidate b_3. Again, by *Tripurari Sharan (2018)*, category-c candidate b_3 receives the open-category seat at college y that is vacated by b_1, ironically profiting from this adjustment. The assignment dictated by the Supreme Court's decision is:

$$\begin{pmatrix} a_1 & a_2 & b_1 & b_2 & b_3 \\ (x,o) & (x,o) & (x,c) & (x,c) & (y,o) \end{pmatrix}.$$

This outcome fails *inter se* merit, because category-c candidate b_2 receives a less-preferred assignment than the assignment of the lower merit ranking category-c candidate b_3. □

5.5. The Case Against the MRC-Based Mechanisms. In Sections 5.3 and 5.4, we have argued that, not only do the allocation mechanisms employed by various Indian institutions have important shortcomings, but also the Supreme Court judgements on these mechanisms have a number of inconsistencies. We conclude this section by arguing that the root cause of these difficulties lies in the excessive reliance on the concept of migration to solve more complex versions of the problem, further exacerbated by the introduction of the status of an MRC as an especially ill-equipped tool to facilitate a solution for applications with heterogenous positions.

Since open positions are allocated prior to the VR-protected positions when all positions are identical, it may be tempting to follow the similar practice when they are also heterogeneous. While a VR-protected candidate may be able to secure an open-category position in this way, it may not necessarily be at his first choice. Consequently, this widespread practice generated the following questions posed in Section 5.1:

(1) Shall these individuals who are promoted to the status of an MRS be allowed to migrate to higher choice jobs, and claim positions set aside for their VR-protected categories?
(2) If they are allowed to migrate, then what happens to the open-category positions they have vacated?

While the first question was answered in the positive by the Supreme Court judgement in *Anurag Patel (2004)*, conflicting decisions were given for the second in the two Supreme Court judgements *Shri Ritesh R. Sah vs Dr. Y.L. Yamul & Ors (1996)*,[39] and *Ramesh Ram (2010)*. However, observe that these questions are not about the fundamentals of the problem, but rather about the mechanics of a specific class of mechanisms.

The root cause of the challenges faced by the MRC-based mechanisms boils down to the following observation: Once an MRC vacates an open position to receive a more-preferred position reserved for his VR-protected category, the next deserving candidate can be,

(1) a member of the general category who is either holding a less-preferred open position from earlier phases, or remains unassigned, or
(2) another MRC who is holding a less-preferred position from earlier phases, or
(3) another member of a reserved category who remains unassigned from earlier phases.

Thus, the widespread practice of the tentative allocation of the open positions in the first phase results in the creation of an artificial interim allocation, one that is often given too much weight despite being a technical construct. This in turn results in awarding the "property rights" of a vacated open position exclusively to the members of a specific category, creating an open invitation for a litigation. This misguided and artificial construction of property rights is the primary source of the dispute in a vast majority of legal conflicts involving MRCs. Indeed, a very similar observation was made by the judges of the Central Administrative Tribunal, Chennai Bench (CAT-CB), in a lower court decision preceding the judgement in *Union of India vs Ramesh Ram & Ors (2010)* by the Supreme Court. The judges of the CAT-CB included the following statement in their ruling:

```
In doing so, the respondents also would notice that the steps taken by them
in accordance with the Rules 16 (3)(-)(5) are redundant once they issue
the result of recruitment in one phase, instead of two as they have become
```

[39] The ruling is available at `https://indiankanoon.org/doc/762690/` (retrieved on 06/01/2022).

```
primary cause for the litigation and avoidable confusion in the minds of the
candidates seeking recruitment.
```

Therefore, the judges have directed the Union of India to announce their outcome in one phase in a manner that respects *inter se* merit, without relying on the artificial concept of migration.[40] However, despite being spot on, this ruling was ignored by the Union of India, and the case moved all the way to the Supreme Court. One possible explanation for the refusal of the Union of India to follow the decision of CAT-CB may be their technical inability to construct a mechanism that complies with the court's order. As we have argued in Section 2, our proposed mechanism 2SMH-DA is uniquely suited for this task.

6. Consequences of the Constitution (103rd Amendment) Act, 2019

In a highly debated reform on the reservation system, the *One Hundred and Third Amendment of the Constitution of India* provides ten percent reservation to the economically weaker sections (EWS) in the general category.[41] While the language of the act is not clear about whether the EWS reservation is intended as a VR policy or an HR policy, a government memorandum dated 01/31/2019 specifies it as the former:[42]

```
7.  ADJUSTMENT AGAINST UNRESERVED VACANCIES:
  A person belonging to EWS cannot be denied the right to compete for
appointment against an unreserved vacancy.  Persons belonging to EWS who
are selected on the basis of merit and not on account of reservation are not
to be counted towards the quota meant for reservation.
```

If the One Hundred and Third Amendment survives the Supreme Court challenge and, implemented as a vertical reservation, it will likely amplify the legal challenges in India due to limitations of MRC-based mechanisms presented in Section 5.5.

It is estimated that, around 26% of the population in India does not belong to the VR-protected categories SC, ST, and OBC.[43] Therefore, in the absence of the new amendment, about 26% of the population belongs to the general category. While the amendment is intended for the economically weaker sections of the general category, according to most

[40]See Appendix C.2 for a comprehensive quote from this case.

[41]The bill of the *One Hundred and Third Amendment of the Constitution of India* was introduced in the Lok Sabha—the lower house of the Parliament of India—on 01/08/2019 as the Constitution (One Hundred and Twenty-fourth Amendment) Bill, 2019. The bill was passed by the Lok Sabha on 01/09/2019, by the Rajya Sabha—the upper house of the Parliament of India—on 01/10/2019, and came into effect on 01/14/2019.

[42]See the Government of India Ministry of Personnel, Public Grievances & Pensions Department of Personnel & Training memorandum No. 36039/1/2019 on Reservation for Economically Weaker Sections (EWSs) in direct recruitment in civil posts and services in the Government of India. This memorandum is available at `https://dopt.gov.in/sites/default/files/ewsf28fT.PDF`, retrieved on 04/14/2019.

[43]See the 01/07/2017-dated *Hindustan Times* story "Quota for economically weak in general category could benefit 190 mn," which is available at `https://tinyurl.com/ycy5uf9m`, retrieved on 02/16/2022.

estimates more than 80% of the members of this group satisfy the eligibility criteria for the EWS reservation.[44] This means, with the introduction of the EWS reservation, the fraction of the population who are ineligible for VR protections reduces to roughly 5-6% of the population. Therefore, the "new general category," those members of the society who are ineligible for VR protections, shrinks to approximately 5-6% of the whole population. This observation, by itself, is not very important. Indeed, inclusion of another VR-protected category has no impact on the analysis of our proposed mechanism 2SMH-DA, presented in Section 3.3. However, the situation is very different for the MRC-based mechanisms discussed and criticized in Sections 5.2-5.5. The reason is that, with the inclusion of EWS to VR-protected categories, the number of VR-protected individuals who are promoted to the status of MRC will increase significantly. Indeed, the fraction of open positions linked to the MRC candidates will likely change from being a minority to a large majority.[45] Therefore, all the problems we emphasized in Section 5.5 can be expected to be amplified, adding to the legal challenges due to these flawed mechanisms.

This observation can be made most clearly for the UPSC mechanism, analyzed in Section 5.3. In Example 3, we have shown that the cut-off score needed for a VR-protected category can be higher under the UPSC mechanism than it is under the general category. The high number of EWS candidates who are expected to be promoted to the status of an MRC candidates, and the ineligibility of EWS candidates for open positions that are vacated from other EWS candidates under *Ramesh Ram (2010)*, means that the minimum cut-off score could easily be higher for EWS candidates than the "new general category" candidates under the UPSC mechanism. Interestingly, this observation has already been made by the officials, who seem to be in search of a solution. The following quote is from a January 2019 *The Hindu* story:[46]

> `While ideally the non-reserved 40% open seats should be open seats based on merit, there are complexities here too. For example, the UPSC accepts a reserved candidate in the civil services examination making it in the general merit list as general only if she has not benefited from reservation in the preliminary, mains, service choice (if one gets a better service, say`

[44] See the 01/28/2019 dated *The Indian Express* story "Whose quota is it anyway? Eligibility criteria for reservation for economically weaker sections will enable the well-off to corner benefits" which is available at `https://indianexpress.com/article/opinion/columns/ews-general-category-quota-sc-st-supreme-court-5557300/` (retrieved on 03/02/2022).

[45] According to the 04/09/2019-dated *India Today* story "Will there be only 31% seats for general category in civil services after new quota?" by Ashok Kumar Upadhyay, an average of 9.15% of all positions allocated by the government's recruiting agency UPSC (including the reserved positions) were allocated to MRC candidates between the years 2008-2017. Since open positions make up 50.5% of all positions, this means roughly 18% of open positions are tentatively allocated to MRC candidates in this period.

[46] See the 01/08/2019 *The Hindu* story "The Hindu Explains: The new 10% quota, its implications, and more," which is available at `https://tinyurl.com/2p8wkwpw` (retrieved on 02/16/2022).

> `IAS or IPS, due to reservation, one is counted as reserved irrespective of one's overall rank) and State cadre choice (if a reserved candidate is in the general merit list but is getting a cadre of her choice as a reserved candidate, she is counted as reserved), say bureaucrats. So, many who are above the general cut-off may still occupy this 10% quota, as they get a better service or cadre in it.`
>
> `A senior IAS officer told` *The Hindu* `that it is possible that a provision will be made for accommodating those who fall below the 10% EWS quota - in case its cut-off is above the general cut-off due to fewer seats - in the open, or general, seats, but this can give rise to litigation.`

We believe our proposed mechanism 2SMH-DA also serves as a natural remedy for this dilemma.

7. Conclusion

Public institutions in India has long struggled with implementing its constitutionally protected VR and HR policies, when either

(1) the two policies are implemented together, or
(2) when positions to allocate are heterogenous.

Many field applications in India such as the allocation of positions for the prestigious All Indian Services (the Indian Administrative Service (IAS), the Indian Police Service (IPS) and the Indian Forest Service (IFS)), and assignment of seats at public educational institutions have both features.

The main challenge for the first complication has been an ambiguity in the legal formulation of the VR policy, originally given in the landmark Supreme Court judgment *Indra Sawhney (1992)*. This ambiguity–which resulted in countless litigations and disruption of recruitment policies in India for the last three decades–has finally been resolved by another important Supreme Court judgment *Saurav Yadav (2020)*. The same judgment also endorsed a mechanism for the joint implementation of VR and HR policies; one that was earlier proposed in Sönmez and Yenmez (2021).

The challenges for the second complication are technically deeper, although a resolution that relies on implementing VR policies across multiple institions with the celebrated deferred acceptance algorithm is straightforward when VR policy is implemented in isolation.[47] However, persons with disabilities is HR protected in India at the federal level,[48] and hence it is vital to implement VR and HR policies together. To the best of our knowledge, no mechanism has been proposed or implemented in India for this general version of the problem.

[47]Such a system is used for allocation of elite high school seats in Chicago (Dur et al., 2020).

[48]Most of the states have awarded HR protections to other groups such as women, ex-servicemen, etc.

In this paper we argue that, even though the recent Supreme Court judgment *Saurav Yadav (2020)* directly concerns applications where all positions are identical, the clarification it has provided for the more general principles originally given in *Indra Sawhney (1992)* offers a natural resolution for the more general version of the problem with heterogenous positions as well. We propose the 2SMH-DA mechanism, which is a refinement of the celebrated individual-proposing deferred acceptance mechanism (Gale and Shapley, 1962) where each institution is endowed with the choice rule 2SMH (Sönmez and Yenmez, 2021). Not only the 2SMH-DA mechanism Pareto dominates any other mechanism that satisfies the Supreme-Court mandated principles in *Saurav Yadav (2020)*, it is the only strategy-proof mechanism that satisfies all these principles. Hence, we believe, 2SMH-DA mechanism is uniquely suited to implement VR and HR policies when positions are heterogenous.

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Appendix A. Additional Terminology and Auxiliary Results

In this appendix, we provide additional terminology and auxiliary lemmas that we use in the proofs.

A.1. Additional Terminology. In this section, we define some terminology to use in the rest of the appendix.

Definition 17. A *job matching* $\mu : \mathcal{I} \to \mathcal{J} \cup \{\emptyset\}$ is a function such that, for every $j \in \mathcal{J}$, $|\mu^{-1}(j)| \leq q_j$.

If $\mu(i) = \emptyset$ for some individual $i \in \mathcal{I}$, then the individual is unmatched.

Definition 18. A *choice rule* is a function $C : 2^{\mathcal{I}} \to 2^{\mathcal{I}}$ such that, for any $I \subseteq \mathcal{I}$,

$$C(I) \subseteq I.$$

Note that, any single-category choice rule (introduced in Definition 14) is a choice rule. Similarly, any aggregate choice rule (introduced in Definition 16) is also a choice rule.

Definition 19. A job matching μ is *stable* with respect to a profile of choice rules $(C_j)_{j\in\mathcal{J}}$ if the following three conditions hold:

(1) *Individual rationality*: For each $i \in \mathcal{I}$, $\mu(i) \succeq_i \emptyset$,
(2) *Job rationality*: For each $j \in \mathcal{J}$, $C_j(\mu^{-1}(j)) = \mu^{-1}(j)$.
(3) *No blocking pairs*: There exist no $i \in \mathcal{I}$ and $j \in \mathcal{J}$ such that $j \succ_i \mu(i)$ and $i \in C_j(\mu^{-1}(j) \cup \{i\})$.

Definition 20. An assignment α is *stable* with respect to a profile of multi-category choice rules $(\vec{C}_j)_{j\in\mathcal{J}}$ if the following three conditions hold:

(1) *Individual rationality*: For each $i \in \mathcal{I}$, $\alpha(i) \succeq_i \emptyset$,
(2) *Job rationality*: For each $j \in \mathcal{J}$ and $v \in \mathcal{V}$, $C_j^v(\alpha^{-1}(j)) = \alpha^{-1}(j, v)$.
(3) *No blocking pairs*: There exist no $i \in \mathcal{I}$ and $j \in \mathcal{J}$ such that $j \succ_i \alpha(i)$ and $i \in \widehat{C}_j(\alpha^{-1}(j) \cup \{i\})$.

Given an assignment $\alpha \in \mathcal{A}$, the *job matching μ induced by assignment α* is constructed as follows: For each $i \in \mathcal{I}$,

$$\mu(i) = \begin{cases} j, & \text{if } \alpha(i) = (j, v) \text{ for some } (j, v) \in \mathcal{J} \times \mathcal{V}, \\ \emptyset, & \text{if } \alpha(i) = \emptyset. \end{cases}$$

It is easy to check that μ is a job matching given that α is an assignment.

Likewise, for an assignment mechanism, there is an induced job matching mechanism where, for every preference profile of individuals, the outcome of the matching mechanism is the matching induced by the outcome of the assignment mechanism for the preference profile.

Given the profile of aggregate choice rules $(\widehat{C}^{2s}_{Ⓜ,j})_{j\in\mathcal{J}}$, consider the individual-proposing deferred acceptance mechanism: At each step of the mechanism, if I is the set of individuals considered for job j, job j tentatively accepts $\widehat{C}^{2s}_{Ⓜ,j}(I)$ without specifying any category and permanently rejects $I \setminus \widehat{C}^{2s}_{Ⓜ,j}(I)$. Call this job matching mechanism the *aggregate meritorious deferred-acceptance mechanism (AM-DA)*, and denote it by $\widehat{\varphi}^{2s}_{Ⓜ}$. For any preference profile $\succ_{\mathcal{I}}\in \mathcal{P}$, the outcome $\widehat{\varphi}^{2s}_{Ⓜ}(\succ_{\mathcal{I}})$ of AM-DA is a job matching.

Strategy-proofness for a job-matching mechanism is defined analogously as the strategy-proofness of an assignment mechanism.

A.2. Choice Rule Properties. In this section, we define choice rule properties and establish some lemmas that we use in our proofs.

Definition 21. (Kelso and Crawford, 1982) A choice rule C satisfies the *substitutes* condition, if, for each $I \subseteq \mathcal{I}$ and $i, i' \in I$,

$$i \in C(I) \text{ and } i' \neq i \implies i \in C(I \setminus \{i'\}).$$

Definition 22. (Aygün and Sönmez, 2013) A choice rule C satisfies the *irrelevance of rejected individuals* condition, if, for each $I \subseteq \mathcal{I}$,

$$i \in I \text{ and } i \notin C(I) \implies C(I \setminus \{i\}) = C(I).$$

For any a job $j \in \mathcal{J}$ and a category $v \in \mathcal{V}$, we next show that the meritorious horizontal choice rule $C^v_{Ⓜ,j}$ satisfies the substitutes condition and the irrelevance of rejected individuals condition.

Lemma 1. *For each $j \in \mathcal{J}$ and $v \in \mathcal{V}$, the single-category choice rule $C^v_{Ⓜ,j}$ satisfies the substitutes condition.*

Proof. Fix a job $j \in \mathcal{J}$ and a category $v \in \mathcal{V}$. Let $C^{v,1}_{Ⓜ,j}(I)$ be the set of individuals who are selected in Step 1, and $C^{v,2}_{Ⓜ,j}(I)$ be the set of individuals who are selected in Step 2 of the choice rule $C^v_{Ⓜ,j}$. Hence, for any $I \subseteq \mathcal{I}$,

$$C^v_{Ⓜ,j}(I) = C^{v,1}_{Ⓜ,j}(I) \cup C^{v,2}_{Ⓜ,j}(I).$$

As the first step of the proof, we will show that the choice rule $C^{v,1}_{Ⓜ,j}$ itself satisfies the substitutes condition.[49]

[49]Fleiner (2001) shows that the greedy rule defined on a matroid is substitutable. In Sönmez and Yenmez (2021) we make the observation that $C^{v,1}_{Ⓜ,j}(I)$ is equivalent to the greedy rule for the transversal matroid on the HR graph of job j and category v with rank function n^v_j. Therefore, Claim 1 also follows from Fleiner (2001). Below we present a direct proof.

Claim 1. *For each $j \in \mathcal{J}$ and $v \in \mathcal{V}$, the single-category choice rule $C^{v,1}_{Ⓜ,j}$ satisfies the substitutes condition.*

Proof. Let $i \in \mathcal{I}$ be such that $i \in C^{v,1}_{Ⓜ,j}(I)$. Let $i' \in I \setminus \{i\}$. We need to show that $i \in C^{v,1}_{Ⓜ,j}(I \setminus \{i'\})$. Towards a contradiction, suppose $i \notin C^{v,1}_{Ⓜ,j}(I \setminus \{i'\})$.

Let H^v_j be the set of HR-protected positions at job j and category v. Let $\mu : I \to H^v_j \cup \{\emptyset\}$ be a trait matching such that,

(1) μ has maximum cardinality in the HR graph for I, and
(2) $\{\ell \in I : \mu(i) \in H^v_j\} = C^{v,1}_{Ⓜ,j}(I)$.

Similarly, let $\nu : I \setminus \{i'\} \to H^v_j \cup \{\emptyset\}$ be a trait matching such that,

(1) ν has maximum cardinality in the HR graph for $I \setminus \{i'\}$, and
(2) $\{\ell \in I \setminus \{i'\} : \nu(i) \in H^v_j\} = C^{v,1}_{Ⓜ,j}(I \setminus \{i'\})$.

By assumption, we have $\mu(i) \in H^v_j$ and $\nu(i) = \emptyset$. Starting with individual i, construct the following sequence of individuals, until another individual who is matched in only one of the two trait matchings μ, ν is encountered:

$$i^1 = i, \quad i^2 = \nu^{-1}(\mu(i^1)), \quad i^3 = \mu^{-1}(\nu(i^2)), \quad i^4 = \nu^{-1}(\mu(i^3)), \quad \ldots$$

Let i^k be the last individual in this sequence. Just as individual i, individual i^k is matched either under trait-matching μ or trait-matching ν but not under both. By construction, any other individual in the sequence is matched under both trait-matchings. Let $I^* = \{i^1, \ldots, i^k\}$ be the set of individuals in the sequence. We have three cases to consider:

Case 1: $|I^*| = k$ is odd.

Construct the following trait-matching $\eta : I \setminus \{i'\} \to H^v_j \cup \{\emptyset\}$ for the set of individuals $I \setminus \{i'\}$: For any $\ell \in I \setminus \{i'\}$,

$$\eta(\ell) = \begin{cases} \mu(\ell), & \text{if } \ell \in I^*, \\ \nu(\ell), & \text{if } \ell \in I \setminus I^*. \end{cases}$$

Observe that $|\eta| = |\nu| + 1$ if $i' \in I^*$ and $|\eta| = |\nu| + 2$ if $i' \notin I^*$. In either scenario $|\eta| > |\nu|$, contradicting the trait-matching ν has maximal cardinality in the HR graph for $I \setminus \{i'\}$.

Case 2: $|I^*| = k$ is even and $\sigma_j(i) > \sigma_j(i^k)$.

Construct the following trait-matching $\eta : I \setminus \{i'\} \to H^v_j \cup \{\emptyset\}$ for the set of individuals $I \setminus \{i'\}$: For any $\ell \in I \setminus \{i'\}$,

$$\eta(\ell) = \begin{cases} \mu(\ell), & \text{if } \ell \in I^*, \\ \nu(\ell), & \text{if } \ell \in I \setminus I^*. \end{cases}$$

Observe that, $|\eta| = |\nu|$. While individual i is matched under η and not under ν, individual i^k is matched under ν and not under η. Any remaining individual who is matched in one trait-matching is also matched under the other one. But since $\sigma_j(i) > \sigma_j(i^k)$, the choice of individual i^k at the expense of individual i under matching ν contradicts the construction of $C^{v,1}_{Ⓜ,j}(I \setminus \{i'\})$.

Case 3: $|I^*| = k$ is even and $\sigma_j(i) < \sigma_j(i^k)$.

Construct the following trait-matching $\eta : I \to H^v_j \cup \{\emptyset\}$ for the set of individuals I: For any $\ell \in I$,

$$\eta(\ell) = \begin{cases} \nu(\ell), & \text{if } \ell \in I^*, \\ \mu(\ell), & \text{if } \ell \in I \setminus I^*. \end{cases}$$

Observe that, $|\eta| = |\mu|$. While individual i^k is matched under η and not under μ, individual i is matched under μ and not under η. Any remaining individual who is matched in one trait-matching is also matched under the other one. But since $\sigma_j(i^k) > \sigma_j(i)$, the choice of individual i at the expense of individual i^k under matching μ contradicts the construction of $C^{v,1}_{Ⓜ,j}(I)$. Since all three cases result in a contradiction, we need to have $i \in C^{v,1}_{Ⓜ,j}(I \setminus \{i'\})$, thus completing the proof of the claim. ■

Let $i \in \mathcal{I}$ be such that $i \in C^{v}_{Ⓜ,j}(I)$. Let $i' \in I \setminus \{i\}$. We need to show that $i \in C^{v}_{Ⓜ,j}(I \setminus \{i'\})$.

If $i \in C^{v,1}_{Ⓜ,j}(I)$, then $i \in C^{v,1}_{Ⓜ,j}(I \setminus \{i'\})$ by Claim 1, which implies $i \in C^{v}_{Ⓜ,j}(I \setminus \{i'\})$ as desired.

If $i \in C^{v,2}_{Ⓜ,j}(I)$ and $i \in C^{v,1}_{Ⓜ,j}(I \setminus \{i'\})$, then we also have $i \in C^{v}_{Ⓜ,j}(I \setminus \{i'\})$ as desired.

Finally, let $i \in C^{v,2}_{Ⓜ,j}(I)$ and $i \notin C^{v,1}_{Ⓜ,j}(I \setminus \{i'\})$. For any $i'' \in I \setminus \{i, i'\}$, the relation $i'' \in C^{v}_{Ⓜ,j}(I)$ implies $i'' \in C^{v,1}_{Ⓜ,j}(I \setminus \{i'\})$ by Claim 1. Therefore,

$$C^{v,1}_{Ⓜ,j}(I) \setminus \{i'\} \subseteq C^{v,1}_{Ⓜ,j}(I \setminus \{i'\}).$$

Moreover, since the function n^v_j is monotone,

$$\left| C^{v,1}_{Ⓜ,j}(I) \right| \geq \left| C^{v,1}_{Ⓜ,j}(I \setminus \{i'\}) \right|.$$

Therefore, if individual i is one of the $\left(q^v_j - \left| C^{v,1}_{Ⓜ,j}(I) \right| \right)$ highest merit ranking individuals in $I \setminus C^{v,1}_{Ⓜ,j}(I)$ under σ_j, then he also has to be one of the $\left(q^v_j - \left| C^{v,1}_{Ⓜ,j}(I \setminus \{i'\}) \right| \right)$ highest merit ranking individuals in $(I \setminus \{i'\}) \setminus \left(C^{v,1}_{Ⓜ,j}(I \setminus \{i'\}) \right)$ under σ_j. Therefore, we have $i \in C^{v,2}_{Ⓜ,j}(I \setminus \{i'\})$, which in turn implies $i \in C^{v}_{Ⓜ,j}(I \setminus \{i'\})$ as desired.

This establishes the desired relation for all three cases, and completes the proof. □

Lemma 2. *For each $j \in \mathcal{J}$ and $v \in \mathcal{V}$, the single-category choice rule $C^v_{Ⓜ,j}$ satisfies the irrelevance of rejected individuals condition.*

Proof. Fix a job $j \in \mathcal{J}$, category $v \in \mathcal{V}$, and $I \subseteq \mathcal{I}$. Let $i \in I$ be such that $i \notin C^{v}_{Ⓜ,j}(I)$. Since $i \notin C^{v,1}_{Ⓜ,j}(I)$ implies $n^v_j(I) = n^v_j(I \setminus \{i\})$, the same individual will be selected in each sub-step of Step 1 of the choice rule $C^{v}_{Ⓜ,j}$ for both sets of individuals I and $I \setminus \{i\}$, and therefore we have $C^{v,1}_{Ⓜ,j}(I \setminus \{i\}) = C^{v,1}_{Ⓜ,j}(I)$. Moreover, an individual i' is one of the $\left(q^v_j - \left|C^{v,1}_{Ⓜ,j}(I)\right|\right)$ highest merit ranking individuals in $I \setminus C^{v,1}_{Ⓜ,j}(I)$ if and only if he is one of the $\left(q^v_j - \left|C^{v,1}_{Ⓜ,j}(I \setminus \{i\})\right|\right) = \left(q^v_j - \left|C^{v,1}_{Ⓜ,j}(I)\right|\right)$ highest merit ranking individuals in $(I \setminus \{i\}) \setminus C^{v,1}_{Ⓜ,j}(I \setminus \{i\})$. Therefore, we also have $C^{v,2}_{Ⓜ,j}(I \setminus \{i\}) = C^{v,2}_{Ⓜ,j}(I)$. Hence, we have $C^{v}_{Ⓜ,j}(I \setminus \{i\}) = C^{v}_{Ⓜ,j}(I)$, establishing that the choice rule $C^{v}_{Ⓜ,j}$ satisfies the irrelevance of rejected individuals condition. □

Definition 23. A choice rule C is *path independent* if, for each $I, I' \subseteq \mathcal{I}$,

$$C(I \cup I') = C\Big(C(I) \cup C(I')\Big).$$

Lemma 3 (Aizerman and Malishevski (1981))**.** *A choice rule satisfies path independence if, and only, if it satisfies both the substitutes condition and the irrelevance of rejected individuals condition.*

Definition 24. A choice rule C satisfies the law of aggregate demand if, for every $I, I' \subseteq \mathcal{I}$

$$I' \supseteq I \implies |C(I')| \geq |C(I)|.$$

Lemma 4. *For each $j \in \mathcal{J}$ and $v \in \mathcal{V}$, the single-category choice rule $C^{v}_{Ⓜ,j}$ satisfies the law of aggregate demand.*

Proof. By construction, for any $I \subseteq \mathcal{I}$, we have $|C^{v}_{Ⓜ,j}(I)| = \min\{r^v_j, |I \cap \mathcal{I}^v|\}$. Let $I' \subseteq I$. Then, we have $\min\{r^v_j, |I' \cap \mathcal{I}^v|\} \leq \min\{r^v_j, |I \cap \mathcal{I}^v|\}$, or equivalently $|C^{v}_{Ⓜ,j}(I)| \leq |C^{v}_{Ⓜ,j}(I')|$. Therefore, $C^{v}_{Ⓜ,j}$ satisfies the law of aggregate demand. □

A.3. Properties of 2-Step Meritorious Horizontal Choice Rule. Fix a job $j \in \mathcal{J}$. In this section, we establish some properties of the 2-step meritorious horizontal choice rule $\vec{C}^{2s}_{Ⓜ,j} = (C^{2s,v}_{Ⓜ,j})_{v \in \mathcal{V}}$ that will be instrumental to prove our main results in Appendix B.

Lemma 5. *Let $I \subseteq \mathcal{I}$ and $i \in I$. If $i \notin \widehat{C}^{2s}_{Ⓜ,j}(I)$, then, for every $v \in \mathcal{V}$,*

$$C^{2s,v}_{Ⓜ,j}(I \setminus \{i\}) = C^{2s,v}_{Ⓜ,j}(I).$$

Proof. First, we establish the desired relation for the open category. Since $C^{2s,o}_{Ⓜ,j} = C^{o}_{Ⓜ,j}$ and $C^{o}_{Ⓜ,j}$ satisfies the irrelevance of rejected individuals condition by Lemma 2, we have $C^{2s,o}_{Ⓜ,j}(I \setminus \{i\}) = C^{2s,o}_{Ⓜ,j}(I)$.

Next we establish the desired relation for any VR-protected category in $\mathcal{R}$. Let $c \in \mathcal{R}$. Then,

$$\begin{aligned} C^{2s,c}_{Ⓜ,j}(I) &= C^{c}_{Ⓜ,j}\Big(\Big(I \setminus C^{o}_{Ⓜ,j}(I)\Big) \cap \mathcal{I}^{c}\Big) \\ &= C^{c}_{Ⓜ,j}\Big(\Big((I \setminus \{i\}) \setminus C^{o}_{Ⓜ,j}(I)\Big) \cap \mathcal{I}^{c}\Big) \\ &= C^{c}_{Ⓜ,j}\Big(\Big((I \setminus \{i\}) \setminus C^{o}_{Ⓜ,j}(I \setminus \{i\})\Big) \cap \mathcal{I}^{c}\Big) \\ &= C^{2s,c}_{Ⓜ,j}(I \setminus \{i\}), \end{aligned}$$

where the first equation holds by definition of $C^{2s,c}_{Ⓜ,j}$, the second equation holds because $i \notin C^{2s,c}_{Ⓜ,j}(I)$ and $C^{c}_{Ⓜ,j}$ satisfies the irrelevance of rejected individuals condition by Lemma 2, the third equation holds because $i \notin C^{2s,o}_{Ⓜ,j}(I) = C^{o}_{Ⓜ,j}(I)$ and $C^{o}_{Ⓜ,j}$ satisfies the irrelevance of rejected individuals condition by Lemma 2, and the last equation holds by definition of $C^{2s,c}_{Ⓜ,j}$. □

The following result is a direct implication of Lemma 5.

Corollary 1. *The aggregate choice rule $\widehat{C}^{2s}_{Ⓜ,j}$ satisfies the irrelevance of rejected individuals condition.*

Lemma 6. *The aggregate choice rule $\widehat{C}^{2s}_{Ⓜ,j}$ satisfies the law of aggregate demand.*

Proof. Let $I', I \subseteq \mathcal{I}$ be such that $I' \supseteq I$. Since $C^{o}_{Ⓜ,j}$ satisfies the law of aggregate demand by Lemma 4,

$$\left|C^{2s,o}_{Ⓜ,j}(I')\right| = \left|C^{o}_{Ⓜ,j}(I')\right| \geq \left|C^{o}_{Ⓜ,j}(I)\right| = \left|C^{2s,o}_{Ⓜ,j}(I)\right|.$$

Furthermore, because $C^{o}_{Ⓜ,j}$ satisfies the substitutes condition by Lemma 1, we have

$$\Big(I \setminus C^{o}_{Ⓜ,j}(I)\Big) \subseteq \Big(I' \setminus C^{o}_{Ⓜ,j}(I')\Big).$$

Consequently, for each $c \in \mathcal{R}$,

$$\left|C^{2s,c}_{Ⓜ,j}(I')\right| = \left|C^{c}_{Ⓜ,j}\Big((I' \setminus C^{o}_{Ⓜ}(I')) \cap \mathcal{I}^{c}\Big)\right| \geq \left|C^{c}_{Ⓜ,j}\Big((I \setminus C^{o}_{Ⓜ}(I)) \cap \mathcal{I}^{c}\Big)\right| = \left|C^{2s,c}_{Ⓜ,j}(I)\right|,$$

because $C^{c}_{Ⓜ,j}$ satisfies the law of aggregate demand (Lemma 4). We conclude that

$$\left|\widehat{C}^{2s}_{Ⓜ,j}(I')\right| = \sum_{v \in \mathcal{V}} \left|C^{2s,v}_{Ⓜ,j}(I')\right| \geq \sum_{v \in \mathcal{V}} \left|C^{2s,v}_{Ⓜ,j}(I)\right| = \left|\widehat{C}^{2s}_{Ⓜ,j}(I)\right|.$$

Therefore, $\widehat{C}^{2s}_{Ⓜ,j}$ satisfies the law of aggregate demand. □

Lemma 7. *The aggregate choice rule $\widehat{C}^{2s}_{Ⓜ,j}$ satisfies the substitutes condition.*

Proof. Let $I \subseteq \mathcal{I}$, $i, i' \in I$, $i \neq i'$, and $i \in \widehat{C}^{2s}_{Ⓜ,j}(I)$. Since $i \in \widehat{C}^{2s}_{Ⓜ,j}(I)$, then either $i \in C^{2s,o}_{Ⓜ,j}(I)$ or $i \in C^{2s,c}_{Ⓜ,j}(I)$ for some $c \in \mathcal{R}$. If $i \in C^{2s,o}_{Ⓜ,j}(I)$, then we have $i \in C^{2s,o}_{Ⓜ,j}(I \setminus \{i'\})$, because $C^{2s,o}_{Ⓜ,j} = C^{o}_{Ⓜ,j}$ satisfies the substitutes condition by Lemma 1. If $i \in C^{2s,c}_{Ⓜ,j}(I)$ for some $c \in \mathcal{R}$, then either (1) $i \in C^{2s,o}_{Ⓜ,j}(I \setminus \{i'\})$ or (2) $i \in (I \setminus \{i'\}) \setminus C^{2s,o}_{Ⓜ,j}(I \setminus \{i'\})$ which implies that $i \in C^{2s,c}_{Ⓜ,j}(I \setminus \{i'\})$ because

(1) $C^{c}_{Ⓜ,j}$ satisfies the substitutes condition,
(2) $I \setminus C^{2s,o}_{Ⓜ,j}(I) \supseteq (I \setminus \{i'\}) \setminus C^{2s,o}_{Ⓜ,j}(I \setminus \{i'\})$, and
(3) $i \in C^{2s,c}_{Ⓜ,j}(I)$.

Therefore, $i \in \widehat{C}^{2s}_{Ⓜ,j}(I \setminus \{i'\})$, and hence $\widehat{C}^{2s}_{Ⓜ,j}$ satisfies the substitutes condition. □

Lemma 8. *The aggregate choice rule $\widehat{C}^{2s}_{Ⓜ,j}$ is path independent.*

Proof. The proof directly follows from Corollary 1, Lemma 3, and Lemma 7. □

Definition 25. Let $I \subseteq \mathcal{I}$ be a set of individuals such that, for every $i \in I$, job j is acceptable to individual i. A multi-category choice rule $\vec{C}_j = (C^v_j)_{v \in \mathcal{V}}$ satisfies *non-wastefulness for I* if, for each $v \in \mathcal{V}$ and $i \in I$,

$$i \notin \widehat{C}_j(I) \;\text{ and }\; |C^v_j(I)| < r^v_j \quad \implies \quad i \notin \mathcal{I}^v.$$

Definition 26. Let $I \subseteq \mathcal{I}$ be a set of individuals such that, for every $i \in I$, job j is acceptable to individual i. A multi-category choice rule $\vec{C}_j = (C^v_j)_{v \in \mathcal{V}}$ satisfies *maximal accommodation of HR protections for I*, if for each $v \in \mathcal{V}$, and $i \in (I \cap \mathcal{I}^v) \setminus \widehat{C}_j(I)$,

$$n^v_j\Big(C^v_j(I)\Big) = n^v_j\Big(C^v_j(I) \cup \{i\}\Big).$$

Definition 27. Let $I \subseteq \mathcal{I}$ be a set of individuals such that, for every $i \in I$, job j is acceptable to individual i. A multi-category choice rule $\vec{C}_j = (C^v_j)_{v \in \mathcal{V}}$ satisfies *no justified envy* if, for each $v \in \mathcal{V}$, $i \in C^v(I)$, and $i' \in \big(I \cap \mathcal{I}^v\big) \setminus \widehat{C}_j(I)$,

$$\sigma_j(i') \;>\; \sigma_j(i) \;\implies\; n^v_j\bigg(\Big(C^v_j(I) \setminus \{i\}\Big) \cup \{i'\}\bigg) < n^v_j\Big(C^v_j(I)\Big).$$

Definition 28. Let $I \subseteq \mathcal{I}$ be a set of individuals such that, for every $i \in I$, job j is acceptable to individual i. A multi-category choice rule $\vec{C}_j = (C^v_j)_{v \in \mathcal{V}}$ satisfies *compliance with VR protections for I* if, for every $c \in \mathcal{R}$ and $i \in C^c_j(I)$,

(1) $|C^o_j(I)| = r^o_j$,
(2) for every $i' \in C^o_j(I)$,

$$\sigma_j(i') < \sigma_j(i) \quad \implies \quad n^o\Big(C^o(I)\Big) > n^o\bigg(\Big(C^o(I) \setminus \{i'\}\Big) \cup \{i\}\bigg), \text{ and}$$

(3) $n^o_j\Big(C^o_j(I) \cup \{i\}\Big) = n^o_j\Big(C^o(I)\Big)$.

Lemma 9 (Sönmez and Yenmez (2021)). *Let $I \subseteq \mathcal{I}$ be a set of individuals such that, for every $i \in I$, job j is acceptable to individual i. A multi-category choice rule $\vec{C}_j$ satisfies (i) non-wastefulness for I, (ii) maximal accommodation of HR protections for I, (iii) no justified envy for I, and (iv) compliance with VR protections for I, if, and only if, $\vec{C}_j(I) = \vec{C}^{2s}_{Ⓜ,j}(I)$.*

Remark 1. Lemma 9 is originally given as Theorem 3 in Sönmez and Yenmez (2021). In their model, there is only one job which is assumed to be acceptable by all individuals. Therefore, the axioms in Sönmez and Yenmez (2021) are stated for every set of individuals $I \subseteq \mathcal{I}$ and Theorem 3 in Sönmez and Yenmez (2021) states that $\vec{C}_j = \vec{C}^{2s}_{Ⓜ,j}$. Since a given job $j \in \mathcal{J}$ may not be acceptable for all individuals in our current setting, we state each axiom used in Lemma 9 for a given set of individuals each of whom finds job j acceptable, and, therefore, state the conclusion as $\vec{C}_j(I) = \vec{C}^{2s}_{Ⓜ,j}(I)$ for any such group of individuals I.

Appendix B. Proofs of Theorem 2 and Theorem 1

We first prove Theorem 2 using several lemmas and then establish Theorem 1.

B.1. Proof of Theorem 2. We provide the proof in several lemmas. Lemmata 10-15 establish that 2SMH-DA satisfies the five axioms, whereas Lemmata 16-20 establish that it is the only assignment mechanism to do so.

Lemma 10. *2SMH-DA satisfies individual rationality.*

Proof. Fix a preference profile $\succ_{\mathcal{I}} = (\succ_i)_{i\in\mathcal{I}} \in \mathcal{P}$. Let assignment $\alpha = \varphi^{2s}_{Ⓜ}(\succ_{\mathcal{I}})$ be the outcome of mechanism 2SMH-DA for $\succ_{\mathcal{I}}$. Let $i \in \mathcal{I}$ be any individual. Since no individual proposes to an unacceptable job under 2SMH-DA, either $\alpha(i) = \emptyset$ or $\alpha(i) = (j, v)$ for a job $j \in \mathcal{J}$ with $j \succ_i \emptyset$ and category $v \in \mathcal{V}$. Therefore, $\alpha(i) \succeq_i \emptyset$, and hence the assignment α satisfies individual rationality. □

Lemma 11. *2SMH-DA satisfies non-wastefulness.*

Proof. Fix a preference profile $\succ_{\mathcal{I}} = (\succ_i)_{i\in\mathcal{I}} \in \mathcal{P}$. Let assignment $\alpha = \varphi^{2s}_{Ⓜ}(\succ_{\mathcal{I}})$ be the outcome of the mechanism 2SMH-DA for $\succ_{\mathcal{I}}$. Suppose that $j \in \mathcal{J}$, $v \in \mathcal{V}$, and $i \in \mathcal{I}$ are such that $j \succ_i \alpha(i)$ and $|\alpha^{-1}(j, v)| < r^v_j$. To show non-wastefulness, we need to establish that $i \notin \mathcal{I}^v$.

Let I be the set of individuals who are considered for job j at the last step of 2SMH-DA. Since $j \succ_i \alpha(i)$ (by assumption) and $\alpha(i) \succeq_i \emptyset$ (by Lemma 10), we have $j \succ_i \emptyset$. Therefore, individual i must have applied to job j at some step of 2SMH-DA, and he must have been rejected by job j prior to the termination of the algorithm. Since $\widehat{C}^{2s}_{Ⓜ,j}$ satisfies path independence by Lemma 8, we have $\widehat{C}^{2s}_{Ⓜ,j}(I \cup \{i\}) = \widehat{C}^{2s}_{Ⓜ,j}(I)$, which in turn implies

$i \notin \widehat{C}^{2s}_{Ⓜ,j}(I \cup \{i\})$. Finally, since $\vec{C}^{2s}_{Ⓜ,j}$ satisfies non-wastefulness by Lemma 9, the relations $i \notin \widehat{C}^{2s}_{Ⓜ,j}(I \cup \{i\})$ and $|C^{2s,v}_{Ⓜ,j}(I \cup \{i\})| = |C^{2s,v}_{Ⓜ,j}(I)| = |\alpha^{-1}(j,v)| < r^v_j$ imply that $i \notin \mathcal{I}^v$. Hence, the assignment α satisfies non-wastefulness. □

Lemma 12. *2SMH-DA satisfies maximal accommodation of HR protections.*

Proof. Fix a preference profile $\succ_{\mathcal{I}} = (\succ_i)_{i\in\mathcal{I}} \in \mathcal{P}$. Let assignment $\alpha = \varphi^{2s}_{Ⓜ}(\succ_{\mathcal{I}})$ be the outcome of mechanism 2SMH-DA for $\succ_{\mathcal{I}}$. Consider a job $j \in \mathcal{J}$, category $v \in \mathcal{V}$, and $i \in \mathcal{I}^v$ such that $j \succ_i \alpha(i)$. To prove that 2SMH-DA satisfies maximal accommodation of HR protections, we need to establish that

$$n^v_j\Big(\alpha^{-1}(j,v) \cup \{i\}\Big) \not> n^v_j\Big(\alpha^{-1}(j,v)\Big).$$

Let I be the set of individuals who are considered for job j at the last step of 2SMH-DA. Since $j \succ_i \alpha(i)$ (by assumption) and $\alpha(i) \succeq_i \emptyset$ (by Lemma 10), we have $j \succ_i \emptyset$. Therefore, individual i must have applied to job j at some step of 2SMH-DA, and he must have been rejected by job j prior to the termination of the algorithm. Since $\widehat{C}^{2s}_{Ⓜ,j}$ satisfies path independence by Lemma 8, we have $\widehat{C}^{2s}_{Ⓜ,j}(I \cup \{i\}) = \widehat{C}^{2s}_{Ⓜ,j}(I)$, which in turn implies $i \notin \widehat{C}^{2s}_{Ⓜ,j}(I \cup \{i\})$. Since $\vec{C}^{2s}_{Ⓜ,j}$ satisfies maximal accommodation of HR protections by Lemma 9, the relation $i \in \Big((I \cup \{i\}) \cap \mathcal{I}^v\Big) \setminus \widehat{C}^{2s}_{Ⓜ,j}\Big(I \cup \{i\}\Big)$ implies

$$n^v_j(\alpha^{-1}(j,v)) = n^v_j\Big(C^{2s,v}_{Ⓜ,j}(I \cup \{i\})\Big) = n^v_j\Big(C^{2s,v}_{Ⓜ,j}(I \cup \{i\}) \cup \{i\}\Big) = n^v_j\Big(\alpha^{-1}(j,v) \cup \{i\}\Big).$$

Therefore,

$$n^v_j\Big(\alpha^{-1}(j,v) \cup \{i\}\Big) \not> n^v_j\Big(\alpha^{-1}(j,v)\Big),$$

which establishes that assignment α satisfies maximal accommodation of HR protections. □

Lemma 13. *2SMH-DA satisfies no justified envy.*

Proof. Fix a preference profile $\succ_{\mathcal{I}} = (\succ_i)_{i\in\mathcal{I}} \in \mathcal{P}$. Let assignment $\alpha = \varphi^{2s}_{Ⓜ}(\succ_{\mathcal{I}})$ be the outcome of mechanism 2SMH-DA for $\succ_{\mathcal{I}}$. Consider $i \in \mathcal{I}$, $j \in \mathcal{J}$, $v \in \mathcal{V}$, and $i' \in \mathcal{I}^v$ such that $\alpha(i) = (j,v)$ and $j \succ_{i'} \alpha(i')$. To prove that 2SMH-DA satisfies no justified envy, we need to establish that,

$$\sigma_j(i) > \sigma_j(i') \quad \text{or} \quad n^v_j\Big(\alpha^{-1}(j,v)\Big) > n^v_j\Big(\alpha^{-1}(j,v) \setminus \{i\}) \cup \{i'\}\Big).$$

If $\sigma_j(i) > \sigma_j(i')$, then we are done. Next assume that $\sigma_j(i) < \sigma_j(i')$.

Let I be the set of individuals who are considered for job j at the last step of 2SMH-DA. Since $j \succ_{i'} \alpha(i')$ (by assumption) and $\alpha(i') \succeq_{i'} \emptyset$ (by Lemma 10), we have $j \succ_{i'} \emptyset$. Therefore, individual i' must have applied to job j at some step of 2SMH-DA, and he must have been rejected by job j prior to the termination of the algorithm. Since $\widehat{C}^{2s}_{Ⓜ,j}$ satisfies path independence by Lemma 8, we have $\widehat{C}^{2s}_{Ⓜ,j}(I \cup \{i'\}) = \widehat{C}^{2s}_{Ⓜ,j}(I)$, which in turn implies $i' \notin \widehat{C}^{2s}_{Ⓜ,j}(I \cup \{i'\})$. Therefore, by Lemma 5, we have

$$C^{2s,v}_{Ⓜ,j}(I \cup \{i'\}) = C^{2s,v}_{Ⓜ,j}(I) = \alpha^{-1}(j,v),$$

which in turn implies $i \in C^{2s,v}_{Ⓜ,j}(I \cup \{i'\})$.

Since $\vec{C}^{2s}_{Ⓜ,j}$ satisfies no justified envy by Lemma 9, the relations $i \in C^{2s,v}_{Ⓜ,j}(I \cup \{i'\}),\ \ i' \in \Big((I \cup \{i'\}) \cap \mathcal{I}^v\Big) \setminus \widehat{C}^{2s}_{Ⓜ,j}(I \cup \{i'\})$, and $\sigma_j(i') > \sigma_j(i)$ imply

$$n^v_j\bigg(\Big(\underbrace{C^{2s,v}_{Ⓜ,j}(I \cup \{i'\})}_{=\alpha^{-1}(j,v)} \setminus \{i\}\Big) \cup \{i'\}\bigg) < n^v_j\Big(\underbrace{C^{2s,v}_{Ⓜ,j}(I \cup \{i'\})}_{=\alpha^{-1}(j,v)}\Big),$$

which establishes that assignment α satisfies no justified envy. □

Lemma 14. *2SMH-DA satisfies compliance with VR protections.*

Proof. Fix a preference profile $\succ_{\mathcal{I}} = (\succ_i)_{i \in \mathcal{I}} \in \mathcal{P}$. Let assignment $\alpha = \varphi^{2s}_{Ⓜ}(\succ_{\mathcal{I}})$ be the outcome of mechanism 2SMH-DA for $\succ_{\mathcal{I}}$. Suppose that $i \in \mathcal{I}$ is such that $\alpha(i) = (j,c)$ for some $j \in \mathcal{J}$ and $c \in \mathcal{R}$. Let I be the set of individuals who are considered for job j at the last step of 2SMH-DA. Then $i \in I$ and $C^{2s,v}_{Ⓜ,j}(I) = \alpha^{-1}(j,v)$ for each $v \in \mathcal{V}$. Since $\vec{C}^{2s}_{Ⓜ,j}$ complies with VR protections by Lemma 9, we have

(1) $|\alpha^{-1}(j,o)| = |C^{2s,o}_{Ⓜ,j}(I)| = r^o_j$,

(2) for each $i' \in \mathcal{I}$ with $\alpha(i') = (j,o)$, we have

$$\sigma_j(i) > \sigma_j(i') \implies n^o_j\Big(C^{2s,o}_{Ⓜ,j}(I)\Big) > n^o_j\Big((C^{2s,o}_{Ⓜ,j}(I) \setminus \{i'\}) \cup \{i\}\Big),$$

or equivalently

$$\sigma_j(i') > \sigma_j(i) \ \text{ or } \ n^o_j\Big(\underbrace{\alpha^{-1}(j,o)}_{=\,C^{2s,o}_{Ⓜ,j}(I)}\Big) > n^o_j\Big((\underbrace{\alpha^{-1}(j,o)}_{=\,C^{2s,o}_{Ⓜ,j}(I)} \setminus \{i'\}) \cup \{i\}\Big), \text{ and}$$

(3) $n^o_j\Big(\alpha^{-1}(j,o)\Big) = n^o_j\Big(C^{2s,o}_{Ⓜ,j}(I)\Big) = n^o_j\Big(C^{2s,o}_{Ⓜ,j}(I) \cup \{i\}\Big) = n^o_j\Big(\alpha^{-1}(j,o) \cup \{i\}\Big)$, which in turn implies

$$n^o_j\Big(\alpha^{-1}(j,o) \cup \{i\}\Big) \not> n^o_j\Big(\alpha^{-1}(j,o)\Big).$$

Therefore, assignment α satisfies compliance with VR protections. □

Lemma 15. *2SMH-DA satisfies strategy-proofness.*

Proof. For any preference profile $\succ_{\mathcal{I}} \in \mathcal{P}$ and individual $i \in \mathcal{I}$, the job matching mechanism AM-DA assigns individual i to a job $j \in \mathcal{J}$ if, and only if, the (assignment) mechanism 2SMH-DA assigns individual i to a pair (j,v) where $v \in \mathcal{V}$ and $i \in \mathcal{I}^v$. Likewise, the job matching mechanism AM-DA keeps individual $i \in \mathcal{I}$ unassigned if, and only if, the mechanism 2SMH-DA keeps individual i unassigned. Hence, for any preference profile $\succ_{\mathcal{I}} \in \mathcal{P}$, the job matching $\widehat{\varphi}^{2s}_{Ⓜ}(\succ_{\mathcal{I}})$ (that is generated by AM-DA) is equal to the job matching that is induced by the assignment $\varphi^{2s}_{Ⓜ}(\succ_{\mathcal{I}})$ (which is generated by 2SMH-DA). Therefore, for any preference profile $\succ_{\mathcal{I}} \in \mathcal{P}$ and individual $i \in \mathcal{I}$,

$$\widehat{\varphi}^{2s}_{Ⓜ}(\succ_{\mathcal{I}})(i) \;\sim_i\; \varphi^{2s}_{Ⓜ}(\succ_{\mathcal{I}})(i).$$

Since the aggregate choice rule $\widehat{C}^{2s}_{Ⓜ,j}$ satisfies the substitutes condition (Lemma 7) and the law of aggregate demand (Lemma 6) for each job $j \in \mathcal{J}$, strategy-proofness of the job matching mechanism AM-DA follows from Theorem 11 in Hatfield and Milgrom (2005). Finally, since each individual $i \in \mathcal{I}$ is indifferent between the outcomes of AM-DA and 2SMH-DA for any given preference profile, strategy-proofness of the job matching mechanism AM-DA implies the strategy-proofness of the assignment mechanism 2SMH-DA as well. □

Lemma 16. *Let α be an assignment that satisfies (i) individual rationality, (ii) non-wastefulness, (iii) maximal accommodation of HR protections, (iv) no justified envy, and (v) compliance with VR protections. Then α is stable with respect to $(\vec{C}^{2s}_{Ⓜ,j})_{j\in\mathcal{J}}$.*

Proof. Let $\succ_{\mathcal{I}} \in \mathcal{P}$ and $\alpha \in \mathcal{A}$ be an assignment that satisfies the axioms in the statement of the lemma. Then α satisfies individual rationality by assumption. To establish stability with respect to $(\vec{C}^{2s}_{Ⓜ,j})_{j\in\mathcal{J}}$, we need to show job rationality and the lack of blocking pairs when each job $j \in \mathcal{J}$ is endowed with the multi-category choice rule $\vec{C}^{2s}_{Ⓜ,j}$.

For each $j \in \mathcal{J}$, define

$$\tilde{I}_j = \{\tilde{i} \in \mathcal{I} \;:\; j \succeq_{\tilde{i}} \alpha(\tilde{i})\}.$$

Since α is individually rational, for every $\tilde{i} \in \tilde{I}_j$, job j is acceptable to individual $\tilde{i}$.

Claim 2. *For each $j \in \mathcal{J}$ and $v \in \mathcal{V}$,*

$$C^{2s,v}_{Ⓜ,j}(\tilde{I}_j) = \alpha^{-1}(j,v),$$

and, for each $j \in \mathcal{J}$,

$$\widehat{C}^{2s}_{Ⓜ,j}(\tilde{I}_j) = \alpha^{-1}(j).$$

Proof. Given a job $j \in \mathcal{J}$ and category $v \in \mathcal{V}$, construct the single-category choice rule C^v_j as follows:

For each $I \subseteq \mathcal{I}$,

$$C^v_j(I) = \begin{cases} C^{2s,v}_{Ⓜ,j}(I), & \text{if } I \neq \tilde{I}_j \\ \alpha^{-1}(j,v), & \text{if } I = \tilde{I}_j \end{cases}$$

Define $\vec{C}_j = (C^v_j)_{v \in \mathcal{V}}$. Observe that, for any two categories $v, v' \in \mathcal{V}$ and $I \subseteq \mathcal{I}$,

$$C^v_j(I) = \alpha^{-1}(j,v) \quad \iff \quad C^{v'}_j(I) = \alpha^{-1}(j,v'), \quad \text{and}$$
$$C^v_j(I) = C^{2s,v}_{Ⓜ,j}(I) \quad \iff \quad C^{v'}_j(I) = C^{2s,v'}_{Ⓜ,j}(I).$$

Therefore, since α is an assignment and $\vec{C}^{2s}_{Ⓜ,j}$ is a multi-category choice rule, $\vec{C}_j = (C^v_j)_{v \in \mathcal{V}}$ is also a multi-category choice rule.

We next show that, for each job $j \in \mathcal{J}$, $\vec{C}_j$ satisfies non-wastefulness for $\tilde{I}_j$, maximal accommodation of HR protections for $\tilde{I}_j$, no justified envy for $\tilde{I}_j$, and compliance with VR protections for $\tilde{I}_j$.

Non-wastefulness for $\tilde{I}_j$: Let $v \in \mathcal{V}$, $i \in \tilde{I}_j \setminus \widehat{C}_j(\tilde{I}_j)$, and $|C^v_j(\tilde{I}_j)| < r^v_j$. By construction, we have $j \succ_i \alpha(i)$ and $|\alpha^{-1}(j,v)| < r^v_j$. Therefore, since assignment α satisfies non-wastefulness, we must have $i \notin \mathcal{I}^v$. Hence, $\vec{C}_j$ satisfies non-wastefulness for $\tilde{I}_j$.

Maximal accommodation of HR protections for $\tilde{I}_j$: Let $v \in \mathcal{V}$ and $i \in (\tilde{I}_j \cap \mathcal{I}^v) \setminus \widehat{C}_j(\tilde{I}_j)$. By construction, we have $j \succ_i \alpha(i)$. Since α satisfies maximal accommodation of HR protections and function n^v_j is monotone, we have

$$n^v_j\Big(\alpha^{-1}(j,v)\Big) = n^v_j\Big(\alpha^{-1}(j,v) \cup \{i\}\Big),$$

or equivalently

$$n^v_j\Big(C^v_j(\tilde{I}_j)\Big) = n^v_j\Big(C^v_j(\tilde{I}_j) \cup \{i\}\Big).$$

Therefore, $\vec{C}_j$ satisfies maximal accommodation of HR protections for $\tilde{I}_j$.

No justified envy for $\tilde{I}_j$: Let $v \in \mathcal{V}$, $i \in C^v_j(\tilde{I}_j) = \alpha^{-1}(j,v)$, and $i' \in (\tilde{I}_j \cap \mathcal{I}^v) \setminus \widehat{C}_j(\tilde{I}_j)$. By construction, we have $j \succ_{i'} \alpha(i')$. Since α satisfies no justified envy, we have

$$\sigma_j(i') > \sigma_j(i) \implies n^v_j\left(\alpha^{-1}(j,v)\right) > n^v_j\left((\alpha^{-1}(j,v) \setminus \{i'\}) \cup \{i\}\right),$$

or equivalently

$$\sigma_j(i') > \sigma_j(i) \implies n^v_j\left(C^v_j(\tilde{I}_j)\right) > n^v_j\left((C^v_j(\tilde{I}_j) \setminus \{i'\}) \cup \{i\}\right).$$

Therefore, $\vec{C}_j$ satisfies no justified envy for $\tilde{I}_j$.

Compliance with VR protections for $\tilde{I}_j$: Let $c \in \mathcal{R}$ and $i \in C^c_j(\tilde{I}_j)$. By construction, $i \in \alpha^{-1}(j,c)$. Since α satisfies condition (1) of the axiom compliance with VR protections, we have $|\alpha^{-1}(j,o)| = r^o_j$, or equivalently

$$|\underbrace{C^o_j(\tilde{I}_j)}_{=\alpha^{-1}(j,o)}| = r^o_j.$$

Furthermore, for each $i' \in C^o_j(\tilde{I}_j)$, we have $\alpha(i') = (j,o)$, and since α satisfies condition (2) of the axiom compliance with VR protections, we have

$$\sigma_j(i') > \sigma_j(i) \text{ or } n^o_j\left(\alpha^{-1}(j,o)\right) > n^o_j\left((\alpha^{-1}(j,o) \setminus \{i'\}) \cup \{i\}\right),$$

or equivalently

$$\sigma_j(i) > \sigma_j(i') \implies n^o_j\Big(\underbrace{C^o_j(\tilde{I}_j)}_{=\alpha^{-1}(j,o)}\Big) > n^o_j\Big((\underbrace{C^o_j(\tilde{I}_j)}_{=\alpha^{-1}(j,o)} \setminus\{i'\}) \cup \{i\}\Big).$$

Finally, since α satisfies condition (3) of the axiom compliance with VR protections, we have $n^o_j\left(\alpha^{-1}(j,o) \cup \{i\}\right) \not> n^o_j\left(\alpha^{-1}(j,o)\right)$, which in turn implies that $n^o_j\left(\alpha^{-1}(j,o) \cup \{i\}\right) = n^o_j\left(\alpha^{-1}(j,o)\right)$ since function n^o_j is monotone. Therefore,

$$n^o_j\Big(\underbrace{C^o_j(\tilde{I}_j)}_{=\alpha^{-1}(j,o)}\Big) = n^o_j\Big(\underbrace{C^o_j(\tilde{I}_j)}_{=\alpha^{-1}(j,o)} \cup\{i\}\Big).$$

Hence, $\vec{C}_j$ complies with VR protections for $\tilde{I}_j$.

We have established that, for any job $j \in \mathcal{J}$, the multi-category choice rule $\vec{C}_j$ satisfies non-wastefulness for $\tilde{I}_j$, maximal accommodation of HR protections for $\tilde{I}_j$, no justified envy for $\tilde{I}_j$, and compliance with VR protections for $\tilde{I}_j$. By Lemma 9, $C^v_j(\tilde{I}_j) = C^{2s,v}_{Ⓜ,j}(\tilde{I}_j)$ for each $v \in \mathcal{V}$. Therefore, for each $j \in \mathcal{J}$ and $v \in \mathcal{V}$,

$$\alpha^{-1}(j,v) = C^v_j(\tilde{I}_j) = C^{2s,v}_{Ⓜ,j}(\tilde{I}_j),$$

and, so, for each $j \in \mathcal{J}$,

$$\alpha^{-1}(j) = \bigcup_{v\in\mathcal{V}} \alpha^{-1}(j,v) = \bigcup_{v\in\mathcal{V}} C^{2s,v}_{Ⓜ,j}(\tilde{I}_j) = \widehat{C}^{2s}_{Ⓜ,j}(\tilde{I}_j),$$

completing the proof of Claim 2. ■

Fix a job $j \in \mathcal{J}$ and category $v \in \mathcal{V}$. By construction, we have $\alpha^{-1}(j) \subseteq \tilde{I}_j$. Therefore, since removing a rejected individual does not change the outcome of $\vec{C}^{2s}_{Ⓜ,j}$ by Lemma 5 and $\alpha^{-1}(j,v) = C^{2s,v}_{Ⓜ,j}(\tilde{I}_j)$ by Claim 2, we have

$$C^{2s,v}_{Ⓜ,j}(\alpha^{-1}(j)) = \alpha^{-1}(j,v).$$

Hence, α satisfies job rationality.

To show that there are no blocking pairs, consider an individual-job pair $(i,j) \in \mathcal{I} \times \mathcal{J}$ such that $j \succ_i \alpha(i)$. By the choice of the pair (i,j), we have $j \succ_i \alpha(i)$, and, therefore, by construction we have $i \in \tilde{I}_j = \{\tilde{i} \in \mathcal{I} \; : \; j \succeq_{\tilde{i}} \alpha(\tilde{i})\}$. By the choice of the pair (i,j), we also have $i \notin \alpha^{-1}(j)$. Since $\alpha^{-1}(j) = \widehat{C}^{2s}_{Ⓜ,j}(\tilde{I}_j)$ by Claim 2 and the aggregate choice rule $\widehat{C}^{2s}_{Ⓜ,j}$

satisfies the irrelevance of rejected individuals condition by Corollary 1, we have

$$\widehat{C}^{2s}_{Ⓜ,j}(\alpha^{-1}(j)) = \widehat{C}^{2s}_{Ⓜ,j}(\alpha^{-1}(j) \cup \{i\}) = \widehat{C}^{2s}_{Ⓜ,j}(\tilde{I}_j) = \alpha^{-1}(j).$$

Since $i \notin \alpha^{-1}(j)$, we get $i \notin \widehat{C}^{2s}_{Ⓜ,j}(\alpha^{-1}(j) \cup \{i\})$. Therefore, there are no blocking pairs.

Hence, we conclude that assignment α is stable with respect to $(\vec{C}^{2s}_{Ⓜ,j})_{j\in\mathcal{J}}$. □

Lemma 17. *Let α be an assignment that is stable with respect to $(\vec{C}^{2s}_{Ⓜ,j})_{j\in\mathcal{J}}$. Then, the job matching μ induced by the assignment α is stable with respect to $(\widehat{C}^{2s}_{Ⓜ,j})_{j\in\mathcal{J}}$.*

Proof. Let assignment α be stable with respect to $(\vec{C}^{2s}_{Ⓜ,j})_{j\in\mathcal{J}}$, and μ be the job matching that is induced by α. Individual rationality of α implies individual rationality of μ. Job rationality of α implies that, for each job $j \in \mathcal{J}$ and category $v \in \mathcal{C}$, we have $C^{2s,v}_{Ⓜ,j}(\alpha^{-1}(j)) = \alpha^{-1}(j,v)$, which implies $\widehat{C}^{2s}_{Ⓜ,j}(\alpha^{-1}(j)) = \alpha^{-1}(j)$. Since $\alpha^{-1}(j) = \mu^{-1}(j)$ by definition, the last equation is equivalent to $\widehat{C}^{2s}_{Ⓜ,j}(\mu^{-1}(j)) = \mu^{-1}(j)$. Hence, μ satisfies job rationality. Finally, consider an individual-job pair $(i,j) \in \mathcal{I} \times \mathcal{J}$ such that $j \succ_i \mu(i)$. Since α has no blocking pairs and $\alpha^{-1}(j) = \mu^{-1}(j)$, we have $i \notin \widehat{C}^{2s}_{Ⓜ,j}(\alpha^{-1}(j) \cup \{i\}) = \widehat{C}^{2s}_{Ⓜ,j}(\mu^{-1}(j) \cup \{i\})$. Therefore, there are no blocking pairs for μ. Hence, μ is stable with respect to $\widehat{C}^{2s}_{Ⓜ,j}$. □

Lemma 18. *Let ϕ be a strategy-proof assignment mechanism and $\widehat{\phi}$ be the job matching mechanism induced from ϕ. Then the job matching mechanism $\widehat{\phi}$ is also strategy-proof.*

Proof. Strategy-proofness of $\widehat{\phi}$ follows from the simple observation that if an individual has a profitable deviation at the induced job matching mechanism for a given preference profile, then she has the same profitable deviation at the assignment mechanism for the same preference profile, because, an individual is indifferent between the categories of any given job but otherwise have strict preferences over the set of jobs and remaining unmatched. □

The following result is a direct implication of Lemma 17 along with Lemma 18.

Corollary 2. *Let ϕ be an assignment mechanism that is strategy-proof and stable with respect to $(\vec{C}^{2s}_{Ⓜ,j})_{j\in\mathcal{J}}$ and $\widehat{\phi}$ be the job matching mechanism induced from ϕ. Then the job matching mechanism $\widehat{\phi}$ is strategy-proof and stable with respect to $(\widehat{C}^{2s}_{Ⓜ,j})_{j\in\mathcal{J}}$.*

The following lemma is a generalization of Theorem 3 in Alcalde and Barberà (1994).[50]

Lemma 19. *Let ϕ be a job matching mechanism that is strategy-proof and stable with respect to $(\widehat{C}^{2s}_{Ⓜ,j})_{j\in\mathcal{J}}$. Then, $\phi = \widehat{\varphi}^{2s}_{Ⓜ}$.*

Proof. Towards a contradiction, suppose that mechanism ϕ is strategy-proof and stable with respect to $(\widehat{C}^{2s}_{Ⓜ,j})_{j\in\mathcal{J}}$, but it differs than the mechanism AM-DA. Then there exists a

[50]There are also similar results in Hirata and Kasuya (2017) and Kominers et al. (2021).

preference profile $\succ_{\mathcal{I}} = (\succ_i)_{i\in\mathcal{I}}$ such that $\phi(\succ_{\mathcal{I}})$ is different than the outcome $\widehat{\varphi}^{2s}_{Ⓜ}(\succ_{\mathcal{I}})$ of AM-DA. Therefore, there exists an individual $i \in \mathcal{I}$ such that $\phi(\succ_{\mathcal{I}})(i) \neq \widehat{\varphi}^{2s}_{Ⓜ}(\succ_{\mathcal{I}})(i)$. Since, for every job j, the choice rule $\widehat{C}^{2s}_{Ⓜ,j}$ satisfies the substitutes condition (Lemma 7) and the law of aggregate demand (Lemma 6), AM-DA produces the individual-optimal stable matching (Hatfield and Milgrom, 2005; Aygün and Sönmez, 2013). Therefore,

$$\widehat{\varphi}^{2s}_{Ⓜ}(\succ_{\mathcal{I}})(i) \ \succ_i \ \phi(\succ_{\mathcal{I}})(i).$$

Since ϕ is individually rational, we have $\phi(\succ_{\mathcal{I}})(i) \succeq_i \emptyset$. Therefore, $\widehat{\varphi}^{2s}_{Ⓜ}(\succ_{\mathcal{I}})(i) \succ_i \emptyset$, which in turn implies $\widehat{\varphi}^{2s}_{Ⓜ}(\succ_{\mathcal{I}})(i) \in \mathcal{J}$. Let $\succ'_i$ be a preference relation where only job $\widehat{\varphi}^{2s}_{Ⓜ}(\succ_{\mathcal{I}})(i)$ is acceptable. Since $\widehat{\varphi}^{2s}_{Ⓜ}(\succ_{\mathcal{I}})$ is stable under $\succ_{\mathcal{I}}$, it is also stable under $(\succ'_i, \succ_{\mathcal{I}\setminus\{i\}})$. For every job j, the choice rule $\widehat{C}^{2s}_{Ⓜ,j}$ satisfies the substitutes condition (Lemma 7) and the law of aggregate demand (Lemma 6). Therefore, by Theorem 8 in Hatfield and Milgrom (2005) (which is also known as the rural hospitals theorem), the job matching $\phi(\succ'_i, \succ_{\mathcal{I}\setminus\{i\}})$ assigns individual i the same number of partners as in job matching $\widehat{\varphi}^{2s}_{Ⓜ}(\succ_{\mathcal{I}})$. Since $\widehat{\varphi}^{2s}_{Ⓜ}(\succ_{\mathcal{I}})(i) \in \mathcal{J}$ and the only acceptable job for i under $\succ'_i$ is $\widehat{\varphi}^{2s}_{Ⓜ}(\succ_{\mathcal{I}})(i)$, we have $\phi(\succ'_i, \succ_{\mathcal{I}\setminus\{i\}})(i) = \widehat{\varphi}^{2s}_{Ⓜ}(\succ_{\mathcal{I}})(i)$. Hence,

$$\underbrace{\phi(\succ'_i, \succ_{\mathcal{I}\setminus\{i\}})(i)}_{=\widehat{\varphi}^{2s}_{Ⓜ}(\succ_{\mathcal{I}})(i)} \ \succ_i \ \phi(\succ_{\mathcal{I}})(i),$$

contradicting strategy-proofness of mechanism ϕ, and completing the proof of the lemma. □

Lemma 20. *Fix a preference profile $(\succ_i)_{i\in\mathcal{I}}$. Let the job matching $\mu = \widehat{\varphi}^{2s}_{Ⓜ}(\succ_{\mathcal{I}})$ be the outcome of the job matching mechanism AM-DA under $(\succ_i)_{i\in\mathcal{I}}$. Let assignment α be such that,*

(1) $\alpha^{-1}(j) = \mu^{-1}(j)$ for each job $j \in \mathcal{J}$, and
(2) α satisfies non-wastefulness, maximal accommodation of HR protections, no justified envy, and compliance with VR protections.

Then $\alpha = \varphi^{2s}_{Ⓜ}(\succ_{\mathcal{I}})$.

Proof. Fix a preference profile $(\succ_i)_{i\in\mathcal{I}}$. Let the job matching $\mu = \widehat{\varphi}^{2s}_{Ⓜ}(\succ_{\mathcal{I}})$ and the assignment α be given as in the statement of the lemma. Observe that, the mechanics of the job matching mechanism AM-DA is identical to the mechanics of the assignment mechanism 2SMH-DA, and the two procedures only differ in the structure of their outcomes. AM-DA only specifies individuals' job assignments. In addition to specifying individuals' job assignments, 2SMH-DA also specifies their category assignments. Since the set of individuals under consideration by any given job $j \in \mathcal{J}$ at the last step of both procedures is $\alpha^{-1}(j) = \mu^{-1}(j)$, all we have to show is, for any job $j \in \mathcal{J}$ and category $v \in \mathcal{V}$,

$$C^{2s,v}_{Ⓜ,j}(\alpha^{-1}(j)) = \alpha^{-1}(j,v).$$

Since the job matching μ satisfies individual rationality, so does the assignment α. Fix a job $j \in \mathcal{J}$. Let $\tilde{I}_j = \{\tilde{i} \in \mathcal{I} : j \succeq_{\tilde{i}} \alpha(\tilde{i})\}$. Since assignment α satisfies individual rationality, non-wastefulness, maximal accommodation of HR protections, no justified envy, and compliance with VR protections, by Claim 2 (in the proof of Lemma 16), we have

$$C^{2s,v}_{Ⓜ,j}(\tilde{I}_j) = \alpha^{-1}(j,v) \qquad \text{for each } v \in \mathcal{V}.$$

Furthermore, since (i) $\alpha^{-1}(j) \subseteq \tilde{I}_j$ by construction and (ii) $\vec{C}^{2s}_{Ⓜ,j}$ does not depend on the rejected individuals by repeated application of Lemma 5, for any category $v \in \mathcal{V}$, we have $C^{2s,v}_{Ⓜ,j}(\alpha^{-1}(j)) = \alpha^{-1}(j,v)$ as desired. Hence, $\alpha = \varphi^{2s}_{Ⓜ}(\succ_{\mathcal{I}})$. □

We are ready to establish that 2SMH-DA is the unique assignment mechanism that satisfies the five axioms. Lemma 16 shows that any assignment mechanism that satisfies the axioms has to be stable with respect to $(\vec{C}^{2s}_{Ⓜ,j})_{j\in\mathcal{J}}$. Corollary 2 shows that for any strategy-proof assignment mechanism which is stable with respect to $(\vec{C}^{2s}_{Ⓜ,j})_{j\in\mathcal{J}}$, the induced matching mechanism is strategy-proof and stable with respect to $(\widehat{C}^{2s}_{Ⓜ,j})_{j\in\mathcal{J}}$. Therefore, for any assignment mechanism that satisfies the axioms, the induced matching mechanism is strategy-proof and stable with respect to $(\widehat{C}^{2s}_{Ⓜ,j})_{j\in\mathcal{J}}$. Lemma 19 shows that AM-DA is the unique job matching mechanism that is strategy-proof and stable with respect to $(\widehat{C}^{2s}_{Ⓜ,j})_{j\in\mathcal{J}}$. Finally, Lemma 20 establishes that for any assignment that satisfies the axioms, the induced job matching is the outcome of AM-DA only if the assignment is the outcome of 2SMH-DA. This concludes the proof of Theorem 2.

B.2. Proof of Theorem 1. Let φ be a mechanism that satisfies individual rationality, non-wastefulness, maximal accommodation of HR protections, no justified envy, and compliance with VR protections. Fix a preference profile $\succ_{\mathcal{I}} \in \mathcal{P}$. Let μ be the job matching that is induced by the assignment $\varphi(\succ_{\mathcal{I}})$.

Since job matching μ is induced by the assignment $\varphi(\succ_{\mathcal{I}})$, for each $i \in \mathcal{I}$, we have

$$\mu(i) \;\sim_i\; \varphi(\succ_{\mathcal{I}})(i).$$

Similarly, since job matching $\widehat{\varphi}^{2s}_{Ⓜ}(\succ_{\mathcal{I}})$ is induced by the assignment $\varphi^{2s}_{Ⓜ}(\succ_{\mathcal{I}})$, for each $i \in \mathcal{I}$, we have

$$\widehat{\varphi}^{2s}_{Ⓜ}(\succ_{\mathcal{I}})(i) \;\sim_i\; \varphi^{2s}_{Ⓜ}(\succ_{\mathcal{I}})(i).$$

By Lemma 16, the assignment $\varphi(\succ_{\mathcal{I}})$ is stable with respect to $(\vec{C}^{2s}_{Ⓜ,j})_{j\in\mathcal{J}}$. Therefore, the job matching μ that is induced by the assignment $\varphi(\succ_{\mathcal{I}})$ is stable with respect to $(\widehat{C}^{2s}_{Ⓜ,j})_{j\in\mathcal{J}}$ by Lemma 17. For any job $j \in \mathcal{J}$ the choice rule $\widehat{C}^{2s}_{Ⓜ,j}$ satisfies the substitutes condition by Lemma 7 and the law of aggregate demand by Lemma 6. Therefore, stability of the job matching μ and Theorem 4 in Hatfield and Milgrom (2005) together imply,[51] for each $i \in \mathcal{I}$,

$$\widehat{\varphi}^{2s}_{Ⓜ}(\succ_{\mathcal{I}})(i) \ \succeq_i \ \mu(i).$$

Hence, for each $i \in \mathcal{I}$,

$$\varphi^{2s}_{Ⓜ}(\succ_{\mathcal{I}})(i) \ \sim_i \ \widehat{\varphi}^{2s}_{Ⓜ}(\succ_{\mathcal{I}})(i) \ \succeq_i \ \mu(i) \ \sim_i \ \varphi(\succ_{\mathcal{I}})(i),$$

establishing that, for any preference profile $\succ_{\mathcal{I}} \in \mathcal{P}$, either

(1) the job matching $\widehat{\varphi}^{2s}_{Ⓜ}(\succ_{\mathcal{I}})$ is equal to the job matching μ that is induced by the assignment $\varphi(\succ_{\mathcal{I}})$, or
(2) the assignment $\varphi^{2s}_{Ⓜ}(\succ_{\mathcal{I}})$ Pareto dominates the assignment $\varphi(\succ_{\mathcal{I}})$.

However, since the assignment $\varphi(\succ_{\mathcal{I}})$ satisfies all five axioms, under the first possibility it must be equal to the assignment $\varphi^{2s}_{Ⓜ}(\succ_{\mathcal{I}})$ by Lemma 20. This establishes that the assignment mechanism $\varphi^{2s}_{Ⓜ}$ Pareto dominates any other assignment mechanism that satisfies the five axioms, concluding the proof of Theorem 1.

Appendix C. Supporting Excerpts from Various Related Judgments

C.1. Saurav Yadav (2020). The following paragraphs in *Saurav Yadav (2020)* clarifies that, any VR-protected individual who deserves an open-category position on the basis of merit should be assigned an open-category position (and not a VR-protected position), including VR-protected individuals who deserve an HR-protected position at open-category. The clarification is important for it removes an ambiguity in the original formulation of VR protections in the landmark Supreme Court judgment *Indra Sawhney (1992)*.

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24.  Thus, according to the second view, different principles must be
adopted at two stages; in that:-.
  (I) At the initial stage when the ‘‘Open or General Category’’ seats are
to be filled, the claim of all reserved category candidates based on merit
must be considered and if any candidates from such reserved categories,
on their own merit, are entitled to be selected against Open or General
Category seats, such placement of the reserved category candidate is not
to affect in any manner the quota reserved for such categories in vertical
reservation.
  (II) However, when it comes to adjustment at the stage of horizontal
reservation, even if, such reserved category candidates are entitled, on
merit, to be considered and accommodated against Open or General Seats, at
that stage the candidates from any reserved category can be adjusted only
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[51]Strictly speaking Hatfield and Milgrom (2005) states this result under an implicit assumption of irrelevance of rejected individuals condition, which is implied by the substitutes condition together with the law of aggregate demand. See Aygün and Sönmez (2013) for further details.

and only if there is scope for their adjustment in their own vertical column of reservation.

Such exercise would be premised on following postulates: -

(A) After the initial allocation of Open General Category seats is completed, the claim or right of reserved category candidates to be admitted in Open General Category seats on the basis of their own merit stands exhausted and they can only be considered against their respective column of vertical reservation.

(B) If there be any resultant adjustment on account of horizontal reservation in Open General Category, only those candidates who are not in any of the categories for whom vertical reservations is provided, alone are to be considered.

(C) In other words, at the stage of horizontal reservation, Open General Category is to be construed as category meant for candidates other than those coming from any of the categories for whom vertical reservation is provided.

25. The second view may lead to a situation where, while making adjustment for horizontal reservation in Open or General Category seats, less meritorious candidates may be adjusted, as has happened in the present matter. Admittedly, the last selected candidates in Open General female category while making adjustment of horizontal reservation had secured lesser marks than the Applicants. The claim of the Applicants was disregarded on the ground that they could claim only and only if there was a vacancy or chance for them to be accommodated in their respective column of vertical reservation.

[...]

31. The second view is thus neither based on any authoritative pronouncement by this Court nor does it lead to a situation where the merit is given precedence. Subject to any permissible reservations i.e. either Social (Vertical) or Special (Horizontal), opportunities to public employment and selection of candidates must purely be based on merit.

Any selection which results in candidates getting selected against Open/General category with less merit than the other available candidates will certainly be opposed to principles of equality. There can be special dispensation when it comes to candidates being considered against seats or quota meant for reserved categories and in theory it is possible that a more meritorious candidate coming from Open/General category may not get selected. But the converse can never be true and will be opposed to the very basic principles which have all the while been accepted by this Court.

Any view or process of interpretation which will lead to incongruity as highlighted earlier, must be rejected.

32. The second view will thus not only lead to irrational results where more meritorious candidates may possibly get sidelined as indicated above but will, of necessity, result in acceptance of a postulate that Open / General seats are reserved for candidates other than those coming from vertical reservation categories. Such view will be completely opposed to the long line of decisions of this Court.

C.2. Ramesh Ram (2010). There are court rulings in India where the judges have observed the failure of the principle of *inter se merit*, one of the failures of the MRC-based mechanisms presented in Section 5.3, and demanded institutions to design mechanisms which avoid this failure. The following quote is given in *Ramesh Ram (2010)*:

Central Administrative Tribunal, Chennai Bench in O.A. No. 690 of 2006 and 775 of 2006 had given the following directions:

‘‘(i) The impugned Rule 16 (2) is declared as valid so long as it is confined to allocation of services and confirms to the ratio of Paras 4 to 6 of Anurag Patel order of the Hon'ble Apex Court.

(ii) The Supplementary List issued by the second respondent to the first respondent dated 3.4.2007 is set aside. This would entail issue of a fresh supplementary result from the reserved list of 64 in such a way that adequate number of OBCs are announced in lieu of the OBCs who have come on merit and brought under General Category. The respondents are directed to rework the result in such a way the select list for all the 457 candidates are announced in one lot providing for 242-general, 117 OBC, 57 SC and 41 ST and also ensure that the candidates in OBC, SC & ST who come on merit and without availing any reservation are treated as general candidates and ensure that on equal number of such reserved candidates who are of merit under General Category, are recruited for OBC, SC & ST respectively and complete the select list for 457. Having done this exercise, the respondents should apply Rule 16 (2) to ensure that allocation of the service is in accordance with rank-cum- preference with priority given to meritorious reserved candidates for service allocation by virtue of Rule 16 (2) which is as per para 5 of Anurag Patel order. The entire exercise, as directed above, should be completed as per the order.

(iii) Applying the ratio of Anurag Patel decision of Hon'ble Apex Court (Paras 6 & 7), if there is need for re-allocation of services, the respondents will take appropriate measures to that extent and complete this process also within two months from the date of receipt of a copy of this order."

The CAT had also issued the following direction as to how the results of the UPSC examinations (2005) should have been announced:

''If the UPSC had followed the decision of the Hon'ble Apex Court cited supra and released the select list in one go for all the 457 vacancies then it would have ensured that the select list contained not only 117 OBCs but also an additional number of OBC candidates by this number, in additional to 117 under 27% reservation, while simultaneously be number of general candidates recruited will be less to the extent of OBCs recruited on merit and included in the general list in the result of Civil Services Examination, 2005. Once this order is met, the successful candidates list will include 242 candidates in the General Category which is inclusive of all those Reserved Category candidates coming on merit plus 117 OBC, 57 SC and 41 ST exclusively from these respective reserved categories by applying relaxed norms for them.. If such a list is subjected to Rule 16(2) of Civil Services Examination, 2005 in present form for making service allocation only and then services are allotted based on Rule 16(2) in this context, then the announcement of recruitment result and allocation services will be both in accordance with law as per various judgments the Hon'ble Apex Court and in accordance with the extent orders issued by the Respondent No.1 and also in keeping with spirit of Rule 16 (2) so that, the meritorious reserved candidates get higher preference service as compared to their lower ranked counter parts in OBC, ST,SC. In doing so, the respondents also would notice that the steps taken by them in accordance with the Rules 16 (3)(-)(5) are redundant once they issue the result of recruitment in one phase, instead of two as they have become primary cause for the litigation and avoidable confusion in the minds of the candidates seeking recruitment.''