Incentivized Kidney Exchange*

Tayfun Sönmez† M. Utku Ünver‡ M. Bumin Yenmez§

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Abstract

Over the last 15 years, kidney exchange has become a mainstream paradigm to increase transplants. However, compatible pairs do not participate, and full benefits from exchange can be realized only if they do. We propose incentivizing compatible pairs to participate in exchange by insuring their patients against future renal failure via increased priority in deceased-donor queue. We analyze equity and welfare benefits of this scheme through a new dynamic continuum model. We calibrate the model with US data and quantify substantial gains from adopting incentivized exchange, both in terms of access to living-donor transplants and reduced competition for deceased-donor transplants.

Keywords: Market design, organ allocation, kidney exchange, equity, efficiency, compatible pairs
JEL codes: D47, C78

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†Boston College, Department of Economics; sonmezt@bc.edu
‡Boston College, Department of Economics and Professorial Research Fellow, Deakin University; unver@bc.edu
§Boston College, Department of Economics; bumin.yenmez@bc.edu

1 Introduction

Transplantation is the best remedy for end-stage renal disease. However, there is a severe shortage of transplant kidneys, which can be harvested from either deceased donors or living donors. As of January 2019, more than 290,000 kidney transplants from deceased donors and more than 150,000 transplants from living donors have been performed in the US. The number of willing living donors has been considerably higher than the number of living-donor transplants performed, yet a large fraction of intended gifts have not materialized due to biological incompatibilities. More than thirty percent of potential living donors are blood-type incompatible, and at least seven percent are tissue-type incompatible, with their intended recipients. Blood-type O patients are especially disadvantaged by these biological barriers because they are only blood-type compatible with blood-type O donors. In contrast, blood-type A patients are blood-type compatible with donors of blood types A and O, blood-type B patients are blood-type compatible with donors of blood types B and O, and blood-type AB patients are blood-type compatible with donors of all blood types.[1] The resulting disadvantage to blood-type O patients is mitigated in deceased-donor transplants by a policy that reserves blood-type O kidneys for blood-type O patients, but a similar policy is not possible for living-donor transplants since a living-donor kidney is typically intended as a gift for a specific patient.

Kidney exchange has become more popular over the last 15 years as a way to circumvent the biological barriers to living-donor transplantation. In its most basic form, a kidney exchange is a swap of donors between two patients who are each incompatible with their own donor but compatible with the other patient's donor. Both donors' intended gifts are realized through the exchange, providing each patient with a transplant. However, blood-type O patients are again less likely to benefit from this transplantation modality. Consider a blood-type O patient unable to receive a transplant from his blood-type-incompatible A donor. The pair can potentially swap donors with a blood-type A patient who has a blood-type O donor. But since the blood-type A patient and blood-type O donor are compatible with each other, they are unlikely to enter a kidney exchange, only arriving when they are tissue-type incompatible. Hence, a large number of "underdemanded" blood-type O patients with blood-type A donors compete for a relatively scarce population of "overdemanded" blood-type A patients with blood-type O donors.[2] These pairs with highly sought-after blood-type O donors become available for exchange only because of a tissue-type incompatibility. Ironically, a biological barrier to transplantation increases in the number of living-donor transplants by facilitating a welfare-increasing utilization of living donors.

Obviously, the competition for an exchange would not be so unfavorable for blood-type O patients with blood-type-incompatible donors if all pairs participated in kidney exchange. Indeed, when a clearinghouse for organized kidney exchange was initially proposed, market designers advo-

[1] For the US, 45.6 percent of the population is blood type O, 37.8 percent is blood type A, 12.6 percent is blood type B, and 4 percent is blood type AB.

[2] Based on 2012-2014 data from the three largest kidney-exchange clearinghouses in the US, the percentage of pairs with blood-type O patients was in the range 58.4–60.7 and the percentage of pairs with blood-type O donors was in the range 30.8–33 (Agarwal et al., 2018).

cated a mechanism where all pairs would participate in exchange, whether they are compatible or not (Roth, Sönmez, and Ünver, 2004). However, since patients with compatible donors do not need an exchange, the practice of kidney exchange evolved mostly without them. Despite the resulting suboptimal utilization of living donors, none of the main kidney exchange systems currently offers any incentives for compatible pairs to participate in exchange.[3] This shortcoming is the motivation of our paper. Our main contribution is introducing and analyzing an incentive scheme that encourages compatible pairs to participate in kidney exchange. The incentive we propose is priority in the deceased-donor queue if the patient needs a repeat transplant, thus serving as insurance against a future kidney failure.[4] This insurance is valuable because the median lifespan of a living-donor transplant kidney is less than 16 years (Matas et al., 2015, conditional on one year survival), and 16 percent of all living-donor transplants fail within the first five years (United States Renal Data System, 2018). While our proposed incentive scheme can be offered to all compatible pairs, we analyze a version where the target group is the set of "overdemanded" pairs. These are compatible pairs where the blood types of the donor and the patient differ and either the donor is of blood type O or the patient is of blood type AB. For these pairs, the donor has a more highly sought-after blood type than the patient, and the patient has more likely to have lower tissue-type incompatibility chance with a random donor than the average patient, and their participation in exchange directly results in an additional transplant to the patient of an "underdemanded" pair.

Our incentive scheme has considerable potential for increasing welfare. Using data from the US, our numerical analysis in Table 4 suggests that incentivized exchange can substantially increase the number of living-donor transplants even for modest participation rates from compatible pairs: In the absence of kidney exchange, 44.7 percent of patients with living donors fail to receive a transplant from their donors. With kidney exchange, the percentage of unutilized living donors falls to 32.1. Increasing participation in the target group further decreases the percentage of unutilized living donors by around 2 percent, and around 180 additional patients receive a transplant each year.[5]

While the primary role of incentivized exchange is to increase the number of living-donor transplants, it also improves equity in access both for living-donor and deceased-donor transplants. Equity in access is one of the main objectives of the Organ Procurement and Transplantation Network

[3]The only exception we are aware of is the single-center kidney exchange program at the Methodist Hospital in San Antonio, where compatible pairs are incentivized with higher quality donors (Bingaman et al., 2012).

[4]A living donor already receives priority in the deceased-donor queue in the event of a kidney failure.

[5]While it is not clear what a "reasonable" participation rate of the target group in incentivized exchange might be, Kranenburg et al. (2006) report a 25 percent rate in a survey from the Netherlands when no additional benefit is offered to the compatible pair, Ratner et al. (2010) report a rate of 50 percent or more when there is some benefit to the patient of the compatible pair based on a survey from New York City, and Hendren et al. (2015) report a participation willingness rate exceeding 90 percent from Canadian population of previous donors and current patients (willingness gets stronger when there is some benefit to the patient of the pair; however, this latter survey had a 42 percent response rate among donors and 100 percent among patients). We observe a positive time trend in these estimates correlated with the wider publicity of kidney exchange practice although their methodologies were somewhat different from each other. These estimates are from self-reported survey studies based on a hypothetical question. The only empirical estimate we could find from a natural experiment is about the substitution willingness between a compatible living donor and an immediate deceased-donor transplant. Choi (2019) reports this rate as 17.3 percent. On the other hand, the rate we really need is the substitution willingness between the direct compatible living donor and another compatible living donor plus future priority on the deceased-donor transplant list.

(OPTN), the body that oversees the allocation of transplant organs in the US.[6] In the November 2016 OPTN report on equity in access, patient blood type was identified as one of the three main contributors to inequity in deceased-donor transplantation.[7] Based on this report (and consistent with our numerical analysis in Section 5), patients of blood types O and B are disadvantaged in the US compared to patients of blood types A and AB. Incentivized exchange improves equity in access for living-donor transplantation, mainly by increasing transplants to blood-type O patients through donor exchanges with incentivized pairs. This in turn improves equity in access for deceased-donor transplantation, since blood-type O patients who benefit from incentivized exchange no longer compete for deceased-donor transplants. Blood types O and B are more common among minority groups, therefore any disadvantage to patients of these blood types leads to inequity in access for patients of minority ethnic backgrounds. As such, incentivized exchange also reduces disparities across ethnic groups. To our knowledge, our proposed policy is the first to enhance both the efficiency and equity of the system.[8]

To analyze the efficiency and equity implications of incentivized exchange, we introduce a new and analytically tractable dynamic large-market model of kidney transplantation.[9] Unlike previous models that focus on a single organ-allocation technology, our model can be used to analyze the impact of various technologies and policies that are often used together and that interact with each other. Through our model, we analytically show that, while all primary technologies increase

[6]Effective March 16, 2000, the US Department of Health and Human Services (HHS) implemented a *Final Rule* establishing a regulatory framework for the structure and operations of the OPTN. The primary goal of the OPTN is "to increase and ensure the effectiveness, efficiency, and equity of organ sharing in the national system of organ allocation," and "to increase the supply of donated organs available for transplantation" (Duda, 2005). Initially, the Final Rule only regulated allocation of deceased-donor organs. Since June 2006, its scope has been extended to include living-donor organs: "Under 42 CFR 121.4(a)(6), the Secretary directs the OPTN to develop policies regarding living organ donors and living organ donor recipients, including policies for the equitable allocation of living-donor organs, in accordance with section 121.8 of the final rule." See `https://www.federalregister.gov/documents/2006/06/16/E6-9401/response-to-solicitation-on-organ-procurement-and-transplantation-network-optn-living-donor` (retrieved on 01/16/2019).

[7]The other two are donor service area and patient PRA, which indicates the likelihood of tissue-type incompatibility for a patient. The primary way to reduce inequity to high PRA patients is to increase the pool size. Hence, incentivized exchange should contribute to this objective as well. Moreover, incentivizing all compatible pairs rather than only overdemanded-type pairs can have a more pronounced benefit for high PRA patients. The report is available at `https://optn.transplant.hrsa.gov/media/2159/equity_in_access_report_201705.pdf` (retrieved on 04/01/2018).

[8]The policy of reserving deceased-donor kidneys for same blood-type patients, called ABO-identical allocation policy, treats different blood types the same way. Therefore, it can be viewed as a procedurally or ex-ante egalitarian policy. However, because of the interaction of the deceased-donor queue with the other transplantation technologies and because the donor-to-patient ratio for different blood types are not the same in practice, the waiting times for deceased-donor kidneys can vary across different blood types. Hence, this policy results in unequal waiting times. Our proposal reduces the difference between the longest and shortest waiting times for different blood type deceased-donor queues relative to the regular exchange (see Table 8). Therefore, the incentivized exchange may be better than the regular exchange for a social planner who exhibits ex-post inequality aversion. See Grant et al. (2012) for a study of an ex-post egalitarian social welfare function.

[9]While traditional matching models mostly focus on static, discrete settings, large-market and continuum models have become increasingly common over the last decade, especially in the context of market-design applications. These models include Kojima and Pathak (2009), Che and Kojima (2010), Lee (2017), Azevedo and Budish (2012), Azevedo and Leshno (2016), Kojima, Pathak, and Roth (2013), Liu and Pycia (2013), and Ashlagi and Roth (2014). See also Ünver (2010), Baccara, Lee, and Yariv (2016), Anderson et al. (2017), and Akbarpour, Li, and Oveis-Gharan (2017) for dynamic matching models.

overall access to kidney transplants, living-donor transplantation and kidney exchange reduce equity in access. In contrast, incentivized exchange increases both overall access and equity in access to transplants.

1.1 Literature Review

Kidney exchange was originally proposed by Rapaport (1986) and later formulated and analyzed as a market-design problem by Roth, Sönmez, and Ünver (2004, 2005b, 2007). The idea of including compatible pairs in kidney exchange was initially evaluated by Ross and Woodle (2000) and further explored by Roth, Sönmez, and Ünver (2004, 2005a), Sönmez and Ünver (2014), and Nicolò and Rodríguez-Álvarez (2017) through a market-design lens. Although this idea was immediately dismissed by Ross and Woodle (2000) on ethical grounds, it has received wider acceptance in recent years (see, for example, Veatch, 2006, Kranenburg et al., 2006, Gentry et al., 2007, Ratner et al., 2010, Steinberg, 2011, Bingaman et al., 2012, and Ferrari et al., 2017). The proof of concept involving exchanges with compatible pairs is documented in Ratner et al. (2010). Moreover, Ratner et al. (2010), Kranenburg et al. (2006), and Hendren et al. (2015) report the results of surveys conducted among patients and living donors. They document that patient and donor attitudes toward exchange are largely positive when the patient benefits from the exchange in some form. From a medical ethics perspective, Veatch (2006) and Steinberg (2011) also advocate for incentives. The literature also explores providing incentives through exchanging the donor of a compatible pair with a younger or genetically closer donor (see Roth, Sönmez, and Ünver, 2004, Ferrari et al., 2017, and Nicolò and Rodríguez-Álvarez, 2017). Bingaman et al. (2012) report the implementation of this proposal by providing younger donors to patients of compatible pairs in a small sample. Such schemes have two drawbacks: they can only incentivize a limited number of compatible pairs, and they can also deter participation by extending waiting times. Our proposal is the first one that can globally and ex ante provide incentives to compatible pairs using tools that are already acceptable within the transplantation community.[10]

2 A Dynamic Model of Kidney Transplantation

Consider patients who need a kidney transplant, where each patient has a blood type $X \in \{O, A, B, AB\}$. Let $\boldsymbol{\pi}_X > 0$ be the inflow rate of first-time blood-type X patients; that is $\boldsymbol{\pi}_X dt$ is the measure of first-time blood-type X patients who arrive to the patient pool in a small time interval dt. Suppose that the expected lifetime while living with kidney disease is distributed

[10]Indeed, our intertemporal insurance scheme found acceptance within the medical community after its associated NSF grant outline and paper draft became publicly available in 2014 (`https://www.nsf.gov/awardsearch/showAward?AWD_ID=1426440`) and 2015 (`https://papers.ssrn.com/sol3/papers.cfm?abstract_id=2551344`), respectively: Gill et al. (2017) make a similar proposal to ours for incentivizing compatible pairs to participate in exchange. In practice, Veale et al. (2017) report three uses of a variant of our proposal, leading to 25 transplants through chain exchanges. This scheme is utilized as follows: The old living donor of a younger patient, who likely will need a kidney transplant in the future, initiates a chain of exchanges in the present by donating her kidney to an incompatible pair. In return, the patient receives priority for a kidney at the end of a similar future chain when his kidney fails. The donor has a short donation window due to her old age, and the insurance scheme helps other pairs receive transplants through chain exchanges in the present, in addition to insuring the potential patient originally paired with the donor.

with a continuous and strictly increasing distribution function $F(\cdot)$ on the interval $[0, T]$, and let $S(\cdot) = 1 - F(\cdot)$ denote the survival function on the same interval. Then the measure of blood-type X patients who are alive after t years is given by $\boldsymbol{\pi}_X S(t)$. In the steady state of this model without transplantation, the total mass of blood-type X patients is $\int_0^T \boldsymbol{\pi}_X S(t)dt$.

2.1 Biological Barriers to Kidney Transplantation

The best remedy for kidney failure is transplantation. There are two potential biological barriers to this procedure. A patient must be both blood-type and tissue-type compatible with a potential donor in order to receive his kidney. Blood-type O donors are blood-type compatible with patients of all four blood types, blood-type A donors are blood-type compatible with patients of blood types A and AB, blood-type B donors are blood-type compatible with patients of blood types B and AB, and blood-type AB donors are blood-type compatible with patients of blood type AB. Hence, other things being equal, blood-type O patients are at a disadvantage in finding a blood-type-compatible kidney donor. We denote blood-type compatibility through a "donation" relation $\triangleright$ over blood types, such that $X \triangleright Y$ means that blood-type X donors are blood-type compatible with blood-type Y patients.

The second potential biological barrier to kidney transplantation is a tissue-type incompatibility. Transplantation is not possible if the patient has preformed antibodies against the donor DNA. To simplify the exposition in the main text, we assume that the probability of tissue-type incompatibility between a donor and a random patient is uniform at θ where $0 < \theta < 1$.[11] Hence, a patient can receive a kidney transplant from a blood-type-compatible donor with probability $(1 - \theta)$. The average θ for kidney deceased-donor queue arrivals in the US, given in Table 1, is in the range 0.047-0.068 according to OPTN data from the last several years.[12]

Average Tissue-type Incompatibility Probability for Entrants in the US										
	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018
Average probability	0.047	0.054	0.056	0.056	0.054	0.062	0.064	0.063	0.068	0.068

Table 1: Average of the reported tissue-type incompatibility probabilities for new entrants and reentrants.

2.2 Deceased-Donor Transplantation

The most common source of transplant kidneys in the US (and in much of the western world) is deceased donors. The United Network for Organ Sharing (UNOS) is the federal contractor in charge of allocating deceased-donor organs in the US, and it uses a points system for kidneys. Since deceased-donor organs perish within a very short period, they are allocated as soon as they are harvested. The UNOS deceased-donor kidney-allocation system has two important features: (1)

[11] We relax this assumption in Appendix F, allowing a non-uniform probability of tissue-type incompatibility between a donor and patients of different tissue types. This appendix also provides micro foundations for our results.

[12] This is according to the average calculated panel reactive antibody (CPRA) data of kidney deceased-donor-queue registrations. See Appendix B for details of this calculation. CPRA measures the percentage of the US population against which the patient would have tissue-type incompatibility, retrieved from `http://optn.transplant.hrsa.gov` (on 10/30/2018) using the "advanced report" option.

the waiting time in the queue is the most significant part of the points system, and (2) kidneys are reserved for patients with the same blood type, with the exception of blood-type A kidneys which can also be allocated to blood-type AB patients.[13]

Reserving organs for same blood-type patients is referred to as an **ABO-identical (ABO-i)** allocation policy. Since blood type AB is relatively rare, ABO-i policy is a good approximation for the allocation of deceased-donor kidneys in the US. Furthermore, given the strong influence of waiting time in the deceased-donor queue, we assume that deceased-donor kidneys are allocated with a **first-in-first-out (FIFO)** matching technology.[14]

Let $\boldsymbol{\delta}_X$ be the inflow rate of blood-type X deceased-donor kidneys. There is a shortage of deceased-donor kidneys throughout the world, so we assume that $\boldsymbol{\delta}_X < \boldsymbol{\pi}_X$ for each blood type X. The median lifespan of a transplanted deceased-donor kidney is almost 12 years (Matas et al., 2015, conditional on one year survival). When a transplanted kidney eventually fails, the recipient reenters the patient pool as if he were a new patient. We assume that repeat patients' survival function is the same as the new entrants'. Let $\phi^{\mathbf{d}}$ be the fraction of the steady-state flow of previous recipients who reenter the patient pool because their transplant failed. Then $\phi^{\mathbf{d}}\boldsymbol{\delta}_X$ is the steady-state flow of blood-type X repeat patients. Therefore, the service rate of blood-type X deceased-donor-queue participants, defined as the supply-to-demand ratio at the deceased-donor queue, is[15]

$$s_X^{\mathbf{d},dec} = \frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X}.$$

2.3 Living-Donor Transplantation

Living-donor transplantation is the second major source of transplant kidneys. In 2017, 29 percent of kidney transplants in the US were from living donors. Let α_X be the fraction of blood-type X patients with a living donor. Living donors are assumed to become available for donation as soon as the patient enters the patient pool. Patients with living donors are referred to as **paired patients**, whereas patients without living donors are referred to as **unpaired patients**. In Section 2.2, we assumed that the inflow of patients is higher than the inflow of deceased-donor kidneys for each blood type. In the rest of the paper, we strengthen this assumption: The inflow

[13] Starting December 2014, deceased-donor kidneys that are in the highest-quality quintile are first offered to the top quintile of patients ranked according to long-term survival chances. A matching protocol similar to first-in-first-out is used to allocate these kidneys to their target group of patients and to allocate other kidneys to all patients. See also Footnote 14.

[14] The use of FIFO in modeling deceased-donor kidney allocation only affects the calculation of waiting times in Appendix C. As long as the steady state is well defined and no transplant kidney is wasted, none of our other results is affected by this assumption. Notably, service rates and the total numbers of transplants are unaffected. Thus, these predictions are valid for the current, modified UNOS deceased-donor kidney allocation scheme.

[15] We will use boldface superscripts next to policy variables to denote different transplantation regimes, in particular

- **d** to denote only deceased-donor transplantation regime,
- **l** to denote deceased-donor and direct living-donor transplantation,
- **e** to denote deceased-donor and living-donor transplantation including regular (i.e., unincentivized) exchange, and
- **i** to denote deceased-donor and living-donor transplantation including both regular and incentivized exchange.

We will use superscript *dec* to denote variables related to patients who receive/wait for deceased-donor transplants and *liv* to denote variables related to patients who receive/wait for living-donor transplants directly or through exchange under the transplant regime in question.

of unpaired patients alone is higher than the inflow of deceased-donor kidneys for each blood type. This assumption easily holds in practice. We assume that each patient has at most one living donor, who is of blood type X with probability $p_X > 0$, and to simplify the analysis we also assume that blood types of the patient and his donor are uncorrelated. Then a blood-type X patient with a living donor is (both blood-type and tissue-type) compatible with his donor with probability $p_X^{\mathbf{l}}$, where

$$p_O^{\mathbf{l}} = (1-\theta)p_O, \qquad p_A^{\mathbf{l}} = (1-\theta)(p_O + p_A),$$
$$p_B^{\mathbf{l}} = (1-\theta)(p_O + p_B), \qquad p_{AB}^{\mathbf{l}} = (1-\theta)(p_O + p_A + p_B + p_{AB}) = (1-\theta).$$

We assume that a patient with a compatible living donor receives the kidney as soon as he needs a transplant without ever entering the deceased-donor queue. Therefore, the service rate of paired blood-type X patients to receive a transplant, defined as the fraction of paired patients who can receive a living-donor transplant, is given as

$$s_X^{\mathbf{l},liv} = p_X^{\mathbf{l}},$$

and the flow of paired blood-type X patients who receive a transplant from their donors is given as

$$\boldsymbol{\lambda}_X = p_X^{\mathbf{l}} \alpha_X \boldsymbol{\pi}_X.$$

Although they last longer than deceased-donor transplants (with a median lifespan of almost 16 years conditional on one year survival, Matas et al., 2015), living-donor transplants can also fail. Let $\phi^{\mathbf{l}} \leq \phi^{\mathbf{d}}$ be the fraction of the flow of steady-state living-donor transplant recipients who reenter the patient pool because their transplants fail. We assume that reentrants no longer have a paired living donor.

For each blood type X, the availability of living donors decreases the flow of arrivals to the deceased-donor queue by $\boldsymbol{\lambda}_X$, but a fraction of that flow, $\phi^{\mathbf{l}}\boldsymbol{\lambda}_X$, reenter due to failure of living-donor transplants. Therefore, the net steady-state flow of patients entering the blood-type X deceased-donor queue is[16]

$$\boldsymbol{\pi}_X^{\mathbf{l}} = \boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X + \phi^{\mathbf{l}}\boldsymbol{\lambda}_X - \boldsymbol{\lambda}_X = \boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X - (1-\phi^{\mathbf{l}})\boldsymbol{\lambda}_X,$$

and the service rate of blood-type X deceased-donor-queue participants is given as

$$s_X^{\mathbf{l},dec} = \frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X^{\mathbf{l}}} = \frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X - (1-\phi^{\mathbf{l}})\boldsymbol{\lambda}_X}.$$

Observe that, for each blood type X, the availability of living-donor transplantation reduces the steady-state flow of patients entering the deceased-donor queue by $(1-\phi^{\mathbf{l}})\boldsymbol{\lambda}_X$. Hence, living-donor transplantation not only benefits the paired patients, but also the unpaired patients by increasing service rates for deceased-donor kidneys.

The total service rate for blood-type X patients is

$$s_X^{\mathbf{l}} = \frac{\boldsymbol{\delta}_X + \boldsymbol{\lambda}_X}{\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X + \phi^{\mathbf{l}}\boldsymbol{\lambda}_X}.$$

[16]Thus, the steady-state flow of patients entering the blood-type X deceased-donor queue takes into account not only the patients without living donors but also the patients with living donors who cannot receive living-donor transplants.

3 Kidney Exchange

While the availability of living-donor transplantation benefits all patient groups, not all willing living donors can donate to their intended recipient. Despite this difficulty, an increasing number of patients with incompatible living donors have been receiving kidney transplants through an exchange with other incompatible patient-donor pairs.

Formally, a two-way **kidney exchange** matches two "mutually compatible" patient-donor pairs: the patient of the first pair is compatible with the donor of the second pair, and the patient of the second pair is compatible with the donor of the first pair. Through an exchange of donors, both patients receive a kidney transplant. While patients with compatible donors can also participate in such exchanges, their participation so far has been very limited since they can directly receive a transplant from their own donors. In this section, we restrict our attention to kidney exchanges between incompatible pairs.

We consider a kidney-exchange program that operates in parallel with the deceased-donor allocation scheme. A patient with a compatible donor immediately receives a transplant from his donor without entering either the deceased-donor queue or the kidney-exchange pool. A patient with an incompatible donor, on the other hand, joins both the deceased-donor queue and the kidney-exchange pool. The patient accepts the first available kidney from either program.

We refer to a pair with a blood-type X patient and a blood-type Y donor as a type $X-Y$ pair. In real life, there are far fewer type $A-O$ pairs in kidney-exchange pools than their reciprocal type $O-A$ pairs. Pairs of the former type are blood-type compatible, so they do not need a kidney exchange unless they are tissue-type incompatible. This is a relatively rare event with small θ. Pairs of the latter type, on the other hand, are blood-type incompatible, and thus they must rely on kidney exchange for a living-donor transplant. This motivates the following assumption:

Assumption 1 *For any two distinct blood types X, Y with $X \triangleright Y$, $\theta p_X \alpha_Y \boldsymbol{\pi}_Y \leq p_Y \alpha_X \boldsymbol{\pi}_X$.*

That is, the inflow of type $X-Y$ pairs (who always join the kidney-exchange pool) is at least as large as the inflow of type $Y-X$ pairs (who only join the kidney-exchange pool when they are tissue-type incompatible). Since θ is small, this assumption easily holds in practice.[17]

To simplify the presentation of our analytical results, we also assume that the inflow of type $B-A$ pairs is at least as large as the inflow of type $A-B$ pairs. This assumption is superfluous and symmetric results hold if the inequality is reversed.

Assumption 2 $p_A \alpha_B \boldsymbol{\pi}_B \geq p_B \alpha_A \boldsymbol{\pi}_A$.

Since there are fewer type $A-O$ pairs in the pool than their reciprocal type $O-A$ pairs (by Assumption 1), it is possible to match every $A-O$ pair as soon as they arrive. While the patient of an arriving type $A-O$ pair is tissue-type incompatible with a θ fraction of the donors of type $O-A$

[17] Based on 2012–2014 data from the three largest kidney-exchange clearinghouses in the US, the percentage of "underdemanded" $O-A$, $O-B$, $O-AB$, $A-AB$, and $B-AB$ pairs was in the range 41.9–43.1 and the percentage of "overdemanded" $A-O$, $B-O$, $AB-O$, $AB-A$, and $AB-B$ pairs was in the range 14–15.2 (Agarwal et al., 2018).

pairs in the pool, he is compatible with a much larger fraction $(1-\theta)$, and mutually compatible with a fraction $(1-\theta)^2$. Similarly, for any two distinct blood types X, Y with $X \triangleright Y$, it is possible to match every type $Y-X$ pair as soon as they arrive. This is also the case for any type $A-B$ pair by Assumption 2. It turns out that this simple observation forms the basis for an optimal exchange mechanism.

Theorem 1 (ABO-identical exchange is optimal) *Suppose Assumptions 1 and 2 hold. Then an exchange policy where an arriving incompatible pair is immediately matched with a mutually compatible pair of its reciprocal type maximizes the measure of transplants to pairs arriving at that moment. Moreover, any such policy maximizes the mass of pairs arriving in an interval that can be matched within that interval.*

Observe that the optimal exchange described in Theorem 1 can accommodate FIFO matching where, whenever possible, an arriving type $X-Y$ pair is matched with the longest waiting mutually compatible pair of its reciprocal type $Y-X$. This is the kidney-exchange mechanism we consider in our theoretical analysis.

The following grouping of transplants is helpful in explaining the effect of exchange on paired patients and highlighting the welfare loss that results from excluding compatible pairs from exchange:

1. For each blood type X, transplants due to tissue-type incompatible pairs of type $X-X$: While these patients are blood-type compatible with their donors, they are tissue-type incompatible. Kidney exchange renders tissue-type incompatibility immaterial for them, since each one can be matched with a mutually compatible pair of identical type as soon as they join the kidney-exchange pool. The resulting net increase in the flow of transplants at steady state is $\theta p_X \alpha_X \boldsymbol{\pi}_X$ for each blood type X.
2. For each pair of distinct blood types X, Y with $X \triangleright Y$, transplants due to tissue-type-incompatible pairs of type $Y-X$: Tissue-type incompatibility becomes immaterial for these patients as well, since they too can be matched with a mutually compatible pair as soon as they join the kidney-exchange pool. The resulting net increase in the flow of transplants at steady state is $2\theta p_X \alpha_Y \boldsymbol{\pi}_Y$, since each tissue-type-incompatible pair of type $Y-X$ facilitates a transplant for a patient of its (blood-type-incompatible) reciprocal type $X-Y$.
3. Transplants due to pairs of types $A-B$ and $B-A$: For patients of type $A-B$ (which has a lower inflow than type $B-A$ by Assumption 2), both blood-type and tissue-type incompatibility become immaterial; they can all be immediately matched with a pair of type $B-A$. The resulting net increase in the flow of transplants at steady state is $2p_B \alpha_A \boldsymbol{\pi}_A$, since each pair of type $A-B$ also facilitates a transplant for a patient of type $B-A$.

Intuitively, kidney exchange eliminates tissue-type incompatibility as a barrier to living-donor transplantation, and, in doing so, it facilitates an additional transplant to a patient with a blood-type-incompatible donor. Furthermore, it also facilitates transplants to all pairs of type $A-B$, and as many transplants to pairs of type $B-A$. For pairs with blood-type O or AB patients, kidney exchange is directly tied to tissue-type incompatibility. Pairs with blood-type AB patients

in the kidney-exchange pool join the pool only because they are tissue-type incompatible with their own donors. Pairs with blood-type O patients in the pool, on the other hand, can only receive a transplant if a mutually compatible pair of their reciprocal type becomes available for exchange through a tissue-type incompatibility. As a result, the effect of kidney exchange on patient groups of blood types O and AB is modest compared to its effect on patient groups of blood types A and B.[18] Indeed, in the absence of tissue-type incompatibility (i.e., for $\theta = 0$), the effect of kidney exchange would be exclusively limited to patients of blood types A and B.

Let $\boldsymbol{\epsilon}_X$ denote the steady-state flow of blood-type X patients who receive a living-donor transplant through kidney exchange. For blood type O and any blood type Y, the flow of tissue-type-incompatible type $Y-O$ pairs is $\theta p_O \alpha_Y \boldsymbol{\pi}_Y$. Therefore, a flow $\theta p_O \alpha_Y \boldsymbol{\pi}_Y$ of type $O-Y$ pairs are matched with type $Y-O$ pairs, and

$$\boldsymbol{\epsilon}_O = \theta p_O(\alpha_O \boldsymbol{\pi}_O + \alpha_A \boldsymbol{\pi}_A + \alpha_B \boldsymbol{\pi}_B + \alpha_{AB} \boldsymbol{\pi}_{AB}).$$

Similarly,

$$\begin{aligned}
\boldsymbol{\epsilon}_A &= \theta p_A(\alpha_A \boldsymbol{\pi}_A + \alpha_{AB} \boldsymbol{\pi}_{AB}) + \theta p_O \alpha_A \boldsymbol{\pi}_A + p_B \alpha_A \boldsymbol{\pi}_A, \\
\boldsymbol{\epsilon}_B &= \theta p_B(\alpha_B \boldsymbol{\pi}_B + \alpha_{AB} \boldsymbol{\pi}_{AB}) + \theta p_O \alpha_B \boldsymbol{\pi}_B + p_B \alpha_A \boldsymbol{\pi}_A, \text{ and} \\
\boldsymbol{\epsilon}_{AB} &= \theta(p_{AB} + p_A + p_B + p_O)\alpha_{AB} \boldsymbol{\pi}_{AB} = \theta \alpha_{AB} \boldsymbol{\pi}_{AB}.
\end{aligned}$$

Since the availability of kidney exchange increases the steady-state flow of living-donor transplants by $\boldsymbol{\epsilon}_X$ for any blood type X, the service rate of paired blood-type X patients receiving a living-donor transplant increases by $\boldsymbol{\epsilon}_X/(\alpha_X \boldsymbol{\pi}_X)$ to

$$s_X^{\mathbf{e},liv} = \frac{\boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X}{\alpha_X \boldsymbol{\pi}_X}.$$

We can summarize the effect of kidney exchange on pairs with living donors as follows:

1. Type $A-B$ and each type $X-Y$ with $Y \triangleright X$: Each patient of these pairs either immediately receives a transplant from his own donor, or immediately receives a transplant through kidney exchange. In either case, they do not wait in the deceased-donor queue or exchange pool.
2. Type $B-A$ and each type $X-Y$ with $X \neq Y$ and $X \triangleright Y$: Patients of these pairs join both the kidney-exchange pool and the deceased-donor queue. They all wait for a transplant and some do not survive.
   (a) For any of these types $X-Y$, if the wait in the kidney-exchange pool is less than the wait in the blood-type X deceased-donor queue, then all surviving pairs of type $X-Y$ receive a transplant through exchange, while none of them receives a transplant from the deceased-donor queue.
   (b) Since all patients of type $X-Y$ receive the first available kidney, the wait in the kidney-exchange pool cannot be more than the wait in the blood-type X deceased-donor queue. If the wait for the kidney-exchange pool $X-Y$ is the same as the blood-type X deceased-donor queue, then patients of type $X-Y$ *pool* with unpaired patients of blood-type X. Among those who survive, some receive a transplant through exchange and the rest receive a transplant

[18] See Theorem 3 in Appendix A for a comparative result regarding the effect of kidney exchange across blood types, formalizing this observation for a homogenous population.

from the deceased-donor queue.

For each blood type X, the availability of kidney exchange along with living-donor transplantation decreases the flow of patients who utilize the deceased-donor queue by $\boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X$; however, a fraction of that flow, $\phi^{\mathbf{l}}(\boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X)$, reenter the patient pool due to the failure of living-donor transplants and they exclusively join the deceased-donor queue. Therefore, the net steady-state flow of patients entering the blood-type X deceased-donor queue is

$$\boldsymbol{\pi}^{\mathbf{e}}_X = \boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X + \phi^{\mathbf{l}}(\boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X) - (\boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X) = \boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X - (1 - \phi^{\mathbf{l}})(\boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X),$$

and the service rate of blood-type X deceased-donor-queue participants increases to

$$s^{\mathbf{e},dec}_X = \frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}^{\mathbf{e}}_X} = \frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X - (1 - \phi^{\mathbf{l}})(\boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X)}.$$

The total service rate for blood-type X patients is

$$s^{\mathbf{e}}_X = \frac{\boldsymbol{\delta}_X + \boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X}{\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X + \phi^{\mathbf{l}}(\boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X)}.$$

4 A New Proposal: Incentivized Exchange

In Section 3, we have seen that kidney exchange increases transplants from living donors. However, real-life applications of kidney exchange are almost exclusively utilized by incompatible pairs, limiting its welfare gains. To see how excluding compatible pairs limits the gains from exchange, it is helpful to focus on the grouping in Section 3.

The critical exchanges are those in group 2: For any distinct blood types X, Y with $X \triangleright Y$, a tissue-type-incompatible type $Y - X$ pair exchanges its donor with the donor of a type $X - Y$ pair. To simplify the discussion, let $X = O$ and $Y = A$. Through this exchange, the patient of the tissue-type-incompatible $A - O$ pair immediately receives a transplant from the donor of the type $O - A$ pair. Hence, with kidney exchange, whether a type $A - O$ pair is tissue-type incompatible or not does not affect when or if its patient receives a transplant. More importantly, this exchange also benefits a type $A - O$ pair. In a way, kidney exchange transforms the "misfortune" of the tissue-type-incompatible type $A - O$ pair to a life-saving opportunity for the type $O - A$ pair. Since the type $A - O$ pair is blood-type compatible, they would not have participated in exchange if they were tissue-type compatible. Kidney exchange not only eliminated tissue-type incompatibility as an obstacle for the transplantation, but it also facilitated a transplant for an additional patient. Put differently, from a social-welfare point of view, there is a welfare loss when a blood-type A patient receives a transplant from a blood-type O donor. When exchange is possible, tissue-type incompatibility avoids this welfare loss and more efficiently utilizes living donor kidneys. But why depend on a relatively rare tissue-type incompatibility to avoid this kind of welfare loss? Any pair of type $A - O$, whether they are tissue-type incompatible or not, can participate in kidney exchange, facilitating a transplant for an additional patient. The challenge here is that a tissue-type-compatible pair of type $A - O$ has no reason to participate in exchange.

As our main contribution, we propose incentivizing compatible pairs to participate in exchange by giving the patient "insurance" against future transplant failure. The insurance takes the form

of prioritizing the patient in the deceased-donor queue in the event of a repeat kidney failure.[19] To incentivize their participation in kidney exchange, these **prioritized reentrants** are placed at the top of the deceased-donor queue altering its FIFO structure.[20] Since in our theoretical dynamic large market model welfare gains only occur with the inclusion of tissue-type-compatible pairs of any type $Y-X$ such that $Y \neq X$ and $X \triangleright Y$ to exchange, we use the incentive scheme only for these pairs. For each such pair of type $Y-X$, let ρ_{Y-X} be the fraction of compatible pairs who are willing to take up the **incentivized-exchange** option. In Appendix D, we use simulations to show that incentivizing compatible pairs of types $A-A$, $B-B$, $O-O$, and $AB-AB$ can also provide non-negligible welfare gains in environments with finite arrivals. Hence, for real-life implementation, we propose providing the incentivized-exchange option to all compatible pairs.

In the absence of incentivized exchange, there is an abundance of type $O-A$ pairs compared to type $A-O$ pairs. For high values of ρ_{A-O}, this may change with incentivized exchange. We assume that compatible pairs only take the incentivized-exchange option if they can immediately participate in exchange, ensuring that type $A-O$ remains "overdemanded."

Assumption 3 *For any two distinct blood types X, Y with $X \triangleright Y$,*

$$[\rho_{Y-X}(1-\theta)+\theta]p_X\alpha_Y\boldsymbol{\pi}_Y \leq p_Y\alpha_X\boldsymbol{\pi}_X.$$

Due to differences in the estimated value of α across different blood types, this assumption holds even when ρ=1. (See Table 2 for an estimate of parameter α for each blood type.) For any two distinct blood types X, Y with $X \triangleright Y$ (and as in the case of kidney exchange), Assumption 3 ensures that it is possible to match every pair of type $Y-X$ at steady state as soon as they arrive. Moreover, replacing Assumption 1 with Assumption 3 ensures that the optimality result in Theorem 1 continues to hold under incentivized exchange. Hence, we proceed with an optimal exchange mechanism where an arriving type $X-Y$ pair, whenever possible, is matched with the longest-waiting mutually compatible pair of its reciprocal type $Y-X$.

Since incentivized exchange simply increases the scope of kidney exchange, the analysis in this section parallels the analysis in Section 3. Recall that in our theoretical analysis the target group for incentivized exchange is tissue-type-compatible pairs of types $A-O$, $B-O$, $AB-O$, $AB-A$, and $AB-B$. Consider such a pair that takes the incentivized-exchange option. The patient of this pair could have received a transplant from his own donor, and, hence, his own transplant

[19] While UNOS is the sole federal contractor in charge of allocating deceased-donor organs in the US, it also administers one of the three main kidney-exchange platforms. There are also several much smaller (typically single-center) systems. If, in the future, UNOS implements our proposed incentivized exchange, it could either offer the incentives exclusively to compatible pairs that register with the UNOS-administered kidney-exchange system, or to any pair that joins kidney exchange regardless of where they register. The first option has the advantage that it could encourage the growth of the UNOS system, which may reduce the inefficiencies that result from several small kidney-exchange platforms. (See Section 8 in Sönmez and Ünver, 2015 for a formal result.) The second option, on the other hand, may result in higher participation in incentivized exchange. In our paper we abstract away from this issue and focus on a single kidney-exchange program.

[20] This incentive can be provided in other ways as well, such as by giving additional points to the patient of the incentivized pair (rather than absolute priority) or even by giving them in-kind incentives. How compatible pairs are incentivized is immaterial to the analysis in the main text, although the analysis on the waiting times in Appendix A.2 relies on prioritizing them in the deceased-donor queue.

does not directly increase the total number of transplants. The increase is due to the patient of the blood-type-incompatible reciprocal type pair, with whom the compatible pair exchanges. Therefore, at steady state, one more transplant occurs for each compatible pair that takes the incentivized-exchange option.

Let $Y - X$ be any type targeted for incentivized exchange. The flow of all $Y - X$ pairs is $p_X \alpha_Y \boldsymbol{\pi}_Y$, the flow of tissue-type-compatible $Y - X$ pairs is $(1-\theta) p_X \alpha_Y \boldsymbol{\pi}_Y$, and the flow of $Y - X$ pairs who take the incentivized-exchange option is $\rho_{Y-X}(1-\theta) p_X \alpha_Y \boldsymbol{\pi}_Y$.

For each blood type X, let $\boldsymbol{\iota}_X$ denote the steady-state flow of the contribution of incentivized exchange on blood-type X living-donor transplants. Each blood-type AB patient with a living donor already receives a living-donor transplant once kidney exchange becomes available, so living-donor transplants to blood-type AB patients do not change with the introduction of incentivized exchange. Therefore,

$$\boldsymbol{\iota}_{AB} = 0.$$

In contrast, patients of the following five types will benefit from incentivized exchange through increased living-donor transplantation: $A - AB$, $B - AB$, $O - A$, $O - B$, and $O - AB$. Living-donor transplants to blood-type A patients with blood-type AB donors increase due to incentivized pairs of type $AB - A$; living-donor transplants to type B patients with AB donors increase due to incentivized pairs of type $AB - B$; and living-donor transplants to blood-type O patients with blood-type-incompatible donors increase due to incentivized pairs of types $A - O$, $B - O$, and $AB - O$. Therefore,

$$\boldsymbol{\iota}_A = \rho_{AB-A}(1-\theta) p_A \alpha_{AB} \boldsymbol{\pi}_{AB}, \qquad \boldsymbol{\iota}_B = \rho_{AB-B}(1-\theta) p_B \alpha_{AB} \boldsymbol{\pi}_{AB}, \text{ and}$$
$$\boldsymbol{\iota}_O = \rho_{A-O}(1-\theta) p_O \alpha_A \boldsymbol{\pi}_A + \rho_{B-O}(1-\theta) p_O \alpha_B \boldsymbol{\pi}_B + \rho_{AB-O}(1-\theta) p_O \alpha_{AB} \boldsymbol{\pi}_{AB}.$$

Since the availability of incentivized exchange weakly increases the steady-state flow of living-donor transplants by $\boldsymbol{\iota}_X$ for any blood type X, the service rate of paired blood-type X patients to receive a living-donor transplant weakly increases by $\boldsymbol{\iota}_X / (\alpha_X \boldsymbol{\pi}_X)$ to

$$s_X^{\mathbf{i},liv} = \frac{\boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X + \boldsymbol{\iota}_X}{\alpha_X \boldsymbol{\pi}_X}.$$

Observe that the service rate of living-donor transplants strictly increases for blood types A, B, and O, and it remains at the maximum rate of one for blood type AB. Moreover, since blood type AB is rare, the flow of arriving pairs is modest for types $AB-A$, $AB-B$, and $AB-O$. Therefore, most of the incentivized pairs are of types $A - O$ or $B - O$, so the primary beneficiaries of incentivized exchange are paired patients of blood type O.

For each blood type X, the availability of incentivized exchange along with living-donor transplantation and kidney exchange decreases the flow of patients waiting in the deceased-donor queue by $\boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X + \boldsymbol{\iota}_X$; however, a fraction of that flow, $\phi^{\mathbf{l}}(\boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X + \boldsymbol{\iota}_X)$, reenter the patient pool due to the failure of living-donor transplants. Therefore, the net steady-state flow of patients entering the blood-type X deceased-donor queue is

$$\boldsymbol{\pi}_X^{\mathbf{i}} = \boldsymbol{\pi}_X + \phi^{\mathbf{d}} \boldsymbol{\delta}_X + \phi^{\mathbf{l}}(\boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X + \boldsymbol{\iota}_X) - (\boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X + \boldsymbol{\iota}_X) = \boldsymbol{\pi}_X + \phi^{\mathbf{d}} \boldsymbol{\delta}_X - (1 - \phi^{\mathbf{l}})(\boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X + \boldsymbol{\iota}_X),$$

and the service rate of blood-type X deceased-donor-queue participants weakly increases to

$$s_X^{\mathbf{i},dec} = \frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X^{\mathbf{i}}} = \frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X - (1-\phi^{\mathbf{l}})(\boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X + \boldsymbol{\iota}_X)}.$$

Then the total service rate for blood-type X patients is

$$s_X^{\mathbf{i}} = \frac{\boldsymbol{\delta}_X + \boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X\boldsymbol{\iota}_X}{\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X + \phi^{\mathbf{l}}(\boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X + \boldsymbol{\iota}_X)}.$$

5 Numerical Model Predictions

In this section, we inspect the predictions of our model by calibrating it with the US patient and donor characteristics. We estimate the proportion of each group served and the number of transplants under various transplantation regimes, including current policy as well as our proposed incentivized exchange.[21] We also run simulations with discrete arrivals using the US population characteristics with either two-way or two-and-three-way exchange technologies. These simulations give us comparable results to the numerical predictions and serve as a robustness check for our theoretical analysis (see Section 5.2 for more on this issue). The simulations are reported in Appendix D.

Benchmark Calibration Parameters				
	O	A	B	AB
ABO-i deceased-donor flows ($\boldsymbol{\delta}_X$) =	5,589	4,343	1,386	395
De-facto deceased-donor flows ($\boldsymbol{\delta}'_X$) =	5,357	4,188	1,548	621
New patient flows ($\boldsymbol{\pi}_X$) =	15,241	10,218	4,626	1,176
Paired-donor blood-type prob. (p_X) =	0.456	0.378	0.126	0.040
Paired-donor fractions (α_X) =	32.84%	24.90%	26.57%	18.93%
Tissue-type incompatibility prob. θ =		0.0679		
Reentry fraction of the recipients $\phi^{\mathbf{l}} = \phi^{\mathbf{d}}$ =		24.10%		
Incentivized-exchange participation fraction (ρ) =		10%, 20%, 30%, 50%, 100%		

Table 2: Benchmark calibration parameters for the numerical policy experiments; time unit is one year. Benchmark survival function $S(t)$ is given in Appendix C.

We report the calibration parameters for our model in Table 2. We explain in Appendices B, C, and D how we obtain these parameters. The second row of Table 2, *de-facto deceased-donor flows* ($\boldsymbol{\delta}'_X$), requires some further explanation. Deceased-donation regulations in the US explicitly dictate that blood-type O and B deceased-donor kidneys are to be transplanted to their respective blood-type patients. However, blood-type O kidneys are occasionally transplanted to blood-type B patients and less frequently to patients of other blood types (see also Subsection 5.1). Moreover, blood-type AB patients occasionally receive kidneys from other blood types. For these reasons, in addition to the strict ABO-i allocation policy, we calculate our model's predictions as if deceased donors arrived according to this observed transplantation allocation across blood types. This is

[21] In Appendix A, we conduct a theoretical analysis of our model and find waiting times for different patient groups. We also estimate the welfare consequences for different policy proposals if living-donation and deceased-donation rates across different blood groups are homogeneous in a population (see Theorems 3 and 4) using service rates. Using the calibrated parameters in this section, we also calculate predicted waiting times in Appendix C.

what we refer to as the de-facto deceased-donor flow for each blood type X, denoted as $\boldsymbol{\delta}'_X$. We conduct all of our analyses using both ABO-i and de-facto deceased-donor flows.

We calculate our model's steady state using these calibration parameters and report outcome variables, such as deceased-donation recipient flows and living-donation recipient flows, $\boldsymbol{\lambda}_X, \boldsymbol{\epsilon}_X, \boldsymbol{\iota}_X$, for different transplantation regimes (see Table 3). We also find the service rate of paired blood-type X patients receiving a living-donor transplant, the service rate of blood-type X deceased-donor-queue participants, and the overall service rate of blood-type X patients receiving either kind of transplant (see Table 4). The overall service rate of blood-type X patients determines what percentage of the patient population receives either kind of transplant and is the ratio of the flow of all transplants to the inflow of all patients, new and reentering, for each blood type X (see the caption of Table 4 for a formal definition). We also pool service rates among all blood types for all patients in this table according to participation in living donation, deceased-donor queue, and either kind of transplant.

5.1 Welfare Consequences

In terms of overall impact, 37.5 percent of patients receive deceased-donor transplants (measured as a fraction of new entrants, $\boldsymbol{\pi}_X$; see the last column in the "Total Transplants" section of Table 3). An additional 15.9 percent receive direct living-donor transplants. An additional 3.6 percent of patients benefit from regular exchange, resulting in 1135 more transplants annually. Our policy proposal, incentivized exchange, helps provide transplants for an additional 0.6 percent of patients (or about 180 additional patients) for each 10 percentage point increase in participation of eligible, compatible pairs annually.[22]

We also consider how each blood type is affected by the introduction of different transplantation regimes. Blood-type B patients are at a disadvantage even when only deceased-donor transplantation is available; they have the lowest service rate of deceased-donor-queue participants (see Table 4). Blood type B is at least twice more common among Asian and African minorities of the US population than among Americans of European descent (see Table 5 of Appendix B). African-Americans are known to be relatively more prone to kidney disease, while the blood-type B deceased-donation rate is not much different from that of other blood types. This explains the lower service rates for Asian and African minorities. Thus, the treatment of blood type B under our proposed policies, as well as blood-type O patients, bears additional importance in equity considerations.

We summarize our main findings regarding the consequences of different transplantation regimes on service rates under the de-facto deceased-donor allocation policy:

[22]We can test the external validity of our model by looking at overall service rates. For example, our model predicts that when regular exchange is available only 50.1 percent of all patients, new arrivals and reentrants, will be served (see the last column of Table 4). This rate is 47.3 percent when only deceased-donor and direct living-donor transplantation is available. Indeed, Scientific Registry of Transplant Patients (SRTR) reports that since 2005, less than 50 percent of the patients of every entering cohort had received transplants as of December 2018. For the SRTR reporting, see `https://www.srtr.org/about-the-data/guide-to-key-transplant-program-metrics/txguidearticles/time-to-transplant/` (retrieved on 12/13/2018).

Numerical Predictions of the Model*: *Patients Receiving Transplant											
		O		*A*		*B*		*AB*		All	
		#	%	#	%	#	%	#	%	#	%
Living-Donor Transplants											
Direct (**l**)		2,127	*14.0%*	1,978	*19.4%*	667	*14.4%*	208	*17.6%*	4,979	*15.9%*
Direct & Exchange (**e**)		2,406	*15.8%*	2,448	*24.0%*	1,038	*22.4%*	223	*18.9%*	6,115	*19.6%*
Direct &	$\rho = 10\%$	2,576	*16.9%*	2,456	*24.0%*	1,040	*22.5%*	223	*18.9%*	6,295	*20.1%*
Regular	$\rho = 20\%$	2,746	*18.0%*	2,464	*24.1%*	1,043	*22.5%*	223	*18.9%*	6,475	*20.7%*
Exchange &	$\rho = 30\%$	2,916	*19.1%*	2,472	*24.2%*	1,045	*22.6%*	223	*18.9%*	6,655	*21.3%*
Incentivized	$\rho = 50\%$	3,255	*21.4%*	2,487	*24.3%*	1,051	*22.7%*	223	*18.9%*	7,016	*22.4%*
Exchange (**i**)	$\rho = 100\%$	4,105	*26.9%*	2,527	*24.7%*	1,064	*23.0%*	223	*18.9%*	7,918	*25.3%*
Total Transplants											
Deceased-Donor Transplantation Only (**d**)											
ABO-i		5,589	*36.7%*	4,343	*42.5%*	1,386	*30.0%*	395	*33.6%*	11,713	*37.5%*
de facto		5,357	*35.2%*	4,188	*41.0%*	1,548	*33.5%*	621	*52.8%*	11,714	*37.5%*
Deceased-/Direct Living-Donor Transplantation (**l**)											
ABO-i		7,716	*50.6%*	6,321	*61.9%*	2,053	*44.4%*	603	*51.2%*	16,693	*53.4%*
de facto		7,484	*49.1%*	6,166	*60.3%*	2,215	*47.9%*	828	*70.4%*	16,694	*53.4%*
Deceased-/Direct Living-Donor Transplantation & Regular Exchange (**e**)											
ABO-i		7,995	*52.5%*	6,791	*66.5%*	2,424	*52.4%*	618	*52.5%*	17,828	*57.0%*
de facto		7,763	*52.5%*	6,636	*66.5%*	2,586	*52.4%*	844	*52.5%*	17,829	*57.0%*
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 10\%$-Incentivized Exchange (**i** $\rho = 10\%$)											
ABO-i		8,165	*53.6%*	6,799	*66.5%*	2,427	*52.5%*	618	*52.5%*	18,008	*57.6%*
de facto		7,933	*53.6%*	6,644	*66.5%*	2,588	*52.5%*	844	*52.5%*	18,009	*57.6%*
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 20\%$-Incentivized Exchange (**i** $\rho = 20\%$)											
ABO-i		8,335	*54.7%*	6,806	*66.6%*	2,429	*52.5%*	618	*52.5%*	18,188	*58.2%*
de facto		8,103	*54.7%*	6,652	*66.6%*	2,591	*52.5%*	844	*52.5%*	18,189	*58.2%*
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 30\%$-Incentivized Exchange (**i** $\rho = 30\%$)											
ABO-i		8,505	*55.8%*	6,814	*66.7%*	2,432	*52.6%*	618	*52.5%*	18,369	*58.8%*
de facto		8,273	*55.8%*	6,660	*66.7%*	2,594	*52.6%*	844	*52.5%*	18,370	*58.8%*
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 50\%$-Incentivized Exchange (**i** $\rho = 50\%$)											
ABO-i		8,844	*58.0%*	6,830	*66.8%*	2,437	*52.7%*	618	*52.5%*	18,729	*59.9%*
de facto		8,613	*58.0%*	6,675	*66.8%*	2,599	*52.7%*	844	*52.5%*	18,730	*59.9%*
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 100\%$-Incentivized Exchange (**i** $\rho = 100\%$)											
ABO-i		9,694	*63.6%*	6,869	*67.2%*	2,450	*53.0%*	618	*52.5%*	19,631	*62.8%*
de facto		9,462	*63.6%*	6,715	*67.2%*	2,612	*53.0%*	844	*52.5%*	19,632	*62.8%*

Table 3: Numerical predictions of the model for the flow of patients receiving transplant (measured in #s per year) for different patient blood types. The percentage on right of each number is the total transplant rate with respect to the new patient flow ($\boldsymbol{\pi}_X$), i.e. $\#/\boldsymbol{\pi}_X$.

Numerical Predictions of the Model: *Service Rate for Transplantation (%)*															
	Living-Donor Trans.					**Deceased-Donor Trans.**					**All Transplants**				
	O	A	B	AB	All	O	A	B	AB	All	O	A	B	AB	All
Deceased-Donor Transplantation Only (**d**)															
ABO-i						33.7	38.5	28.0	31.1	34.4	33.7	38.5	28.0	31.1	34.4
de facto						32.4	37.3	31.0	46.8	34.4	32.4	37.3	31.0	46.8	34.4
Deceased-/Direct Living-Donor Transplantation (**l**)															
ABO-i	42.5	77.7	54.2	93.2	55.3	37.3	44.5	31.1	35.5	38.7	45.1	53.8	40.1	45.6	47.3
de facto	42.5	77.7	54.2	93.2	55.3	35.9	43.1	34.5	53.1	38.7	43.9	52.7	42.9	60.2	47.3
Deceased-/Direct Living-Donor Transplantation & Regular Exchange (**e**)															
ABO-i	48.1	96.2	84.4	100.0	67.9	37.9	46.2	33.2	35.8	39.8	46.6	57.3	46.5	46.6	50.1
de facto	48.1	96.2	84.4	100.0	67.9	36.4	44.7	36.8	53.7	39.8	45.4	56.2	49.3	61.1	50.1
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 10\%$-Incentivized Exchange (**i** $\rho = 10\%$)															
ABO-i	51.5	96.5	84.6	100.0	69.9	38.2	46.2	33.2	35.8	40.0	47.4	57.3	46.6	46.6	50.6
de facto	51.5	96.5	84.6	100.0	69.9	36.8	44.7	36.8	53.7	40.0	46.2	56.2	49.3	61.1	50.6
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 20\%$-Incentivized Exchange (**i** $\rho = 20\%$)															
ABO-i	54.9	96.8	84.8	100.0	71.9	38.5	46.2	33.3	35.8	40.2	48.3	57.4	46.6	46.6	51.0
de facto	54.9	96.8	84.8	100.0	71.9	37.1	44.8	36.8	53.7	40.2	47.1	56.3	49.3	61.1	51.0
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 30\%$-Incentivized Exchange (**i** $\rho = 30\%$)															
ABO-i	58.3	97.1	85.0	100.0	73.9	38.9	46.3	33.3	35.8	40.3	49.2	57.5	46.7	46.6	51.5
de facto	58.3	97.1	85.0	100.0	73.9	37.4	44.8	36.8	53.7	40.3	48.0	56.3	49.4	61.1	51.5
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 50\%$-Incentivized Exchange (**i** $\rho = 50\%$)															
ABO-i	65.0	97.8	85.5	100.0	77.9	39.6	46.3	33.3	35.8	40.7	50.9	57.6	46.7	46.6	52.4
de facto	65.0	97.8	85.5	100.0	77.9	38.1	44.8	36.8	53.7	40.7	49.7	56.4	49.5	61.1	52.4
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 100\%$-Incentivized Exchange (**i** $\rho = 100\%$)															
ABO-i	82.0	99.3	86.5	100.0	88.0	41.5	46.5	33.4	35.8	41.7	55.1	57.9	47.0	46.6	54.5
de facto	82.0	99.3	86.5	100.0	88.0	39.9	45.0	36.9	53.7	41.7	54.0	56.7	49.7	61.1	54.5

Table 4: Numerical predictions of the model for each regime $\mathbf{t} \in \{\mathbf{d}, \mathbf{l}, \mathbf{e}, \mathbf{i}\}$ service rate of paired patients to receive living-donor transplants ($s_X^{\mathbf{t},liv} = \frac{\boldsymbol{\lambda}_X+\boldsymbol{\epsilon}_X+\boldsymbol{\iota}_X}{\alpha_X \boldsymbol{\pi}_X}$ for each blood type X and $s^{\mathbf{t},liv} = \frac{\sum_X(\boldsymbol{\lambda}_X+\boldsymbol{\epsilon}_X+\boldsymbol{\iota}_X)}{\sum_X(\alpha_X \boldsymbol{\pi}_X)}$ for total), service rate of deceased-donor-queue participants ($s_X^{\mathbf{t},dec} = \frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X+\phi^{\mathbf{d}}\boldsymbol{\delta}_X-(1-\phi^{\mathbf{l}})(\boldsymbol{\lambda}_X+\boldsymbol{\epsilon}_X+\boldsymbol{\iota}_X)}$ for each X and $s^{\mathbf{t},dec} = \frac{\sum_X \boldsymbol{\delta}_X}{\sum_X\left(\boldsymbol{\pi}_X+\phi^{\mathbf{d}}\boldsymbol{\delta}_X-(1-\phi^{\mathbf{l}})(\boldsymbol{\lambda}_X+\boldsymbol{\epsilon}_X+\boldsymbol{\iota}_X)\right)}$ for total), and overall service rate ($s_X^{\mathbf{t}} = \frac{\boldsymbol{\delta}_X+\boldsymbol{\lambda}_X+\boldsymbol{\epsilon}_X+\boldsymbol{\iota}_X}{\boldsymbol{\pi}_X+\phi^{\mathbf{d}}\boldsymbol{\delta}_X+\phi^{\mathbf{l}}(\boldsymbol{\lambda}_X+\boldsymbol{\epsilon}_X+\boldsymbol{\iota}_X)}$ for each X and $s^{\mathbf{t}} = \frac{\sum_X(\boldsymbol{\delta}_X+\boldsymbol{\lambda}_X+\boldsymbol{\epsilon}_X+\boldsymbol{\iota}_X)}{\sum_X\left(\boldsymbol{\pi}_X+\phi^{\mathbf{d}}\boldsymbol{\delta}_X+\phi^{\mathbf{l}}(\boldsymbol{\lambda}_X+\boldsymbol{\epsilon}_X+\boldsymbol{\iota}_X)\right)}$ for total) where $\boldsymbol{\lambda}_X = \boldsymbol{\epsilon}_X = \boldsymbol{\iota}_X = 0$ in regime **d** (deceased-donor transplantation only), $\boldsymbol{\epsilon}_X = \boldsymbol{\iota}_X = 0$ in regime **l** (deceased-donor/direct living-donor transplantation), and $\boldsymbol{\iota}_X = 0$ in regime **e** (deceased-donor/direct living-donor transplantation and regular exchange). Every rate is reported in percents. For the de-facto deceased-donor allocation policy, $\boldsymbol{\delta}_X$ is replaced by $\boldsymbol{\delta}'_X$ in each formula.

- The largest benefits from deceased-donor and direct living-donor transplantation go to blood-type AB patients, with blood-type A patients, blood-type O patients, and blood-type B patients receiving successively smaller welfare gains. While the overall service rate is 60.2 percent for blood-type AB patients and 52.7 percent for blood-type A patients, the overall service rate from these two modalities falls to less than 44 percent for blood-type O patients and to less than 43 percent for blood-type B patients (see the last five columns of Table 4).
- At the margin, blood-type B patients benefit the most from regular exchange. Blood-type A patients benefit second most, while blood-type O and blood-type AB patients benefit the least. An additional 6.4 percent of all blood-type B patients receive a kidney due to regular exchange, taking into consideration the increase in reentries to the patient pool caused by the additional steady-state living-donor transplants. The corresponding service-rate increase for blood type A is 3.5 percent, and 1.5 and 0.9 respectively for blood-type O and blood-type AB patients. The widest service-rate gap, the gap between the service rates of blood-type O and AB patients as a result of deceased-donor/direct living-donor transplantation and regular exchange, is 15.7 percent.
- Blood type O patients are the main beneficiaries of incentivized exchange. For each $\Delta\rho = 10$ percent participation increase, incentivized exchange provides kidney to an additional $0.8 - 0.9$ percent of all blood-type O patients. The overall service rates are unaffected for blood-type AB, and the increase is modest for blood types A and B. Thus, the widest service-rate gap, the gap between those of blood-type AB and O patients, decreases by $0.8 - 0.9$ percent for each $\Delta\rho = 10$ percent increase.
- Service rates for deceased-donor-queue participants slightly increase with the introduction of new exchange technologies overall and across blood types. Although additional transplants under these technologies cause an increased number of patients to reenter the patient pool at steady state, even a higher number of additional paired patients receive living-donor transplants and drop out of competition for deceased-donor transplants. The service rate of deceased-donor-queue participants increases from 38.7 percent under deceased-donor/direct living-donor transplantation to 39.8 percent with the availability of regular exchange. With each $\Delta\rho = 10$ percent participation increase in incentivized exchange, this rate increases by about 0.2 percent. Among unpaired patient groups, blood-type O patients benefit the most from incentivized exchange (see the sixth–ninth columns of Table 4).

Thus, incentivized exchange not only helps all patient groups by increasing transplants, but it also mitigates the inequities in access to deceased-donor and living-donor transplantation due to medical incompatibilities (as in the case of blood type O patients) and patient-arrival asymmetries (as in the case of blood type B patients).

5.2 Stress Testing The Model

The magnitudes of our findings depend on a few key parameters. One of these is the tissue-type incompatibility probability θ for arriving patients.[23] It is important to note that, this probability is higher for the entire pool of patients. In real life, two separate mechanisms may cause the tissue-type incompatibility probability for the waiting mass of the patients to go up while they are waiting for a transplant. First, patients with higher intrinsic tissue-type incompatibility, who are referred to as highly sensitized patients, are less likely to receive transplants as they are less likely to be compatible with donors. Second, patients often need blood transfusions while they are waiting for a transplant. Blood transfusions cause foreign tissue-type antigens to enter the body. As a result, new antibodies are formed against these tissue types, causing the patient's intrinsic tissue-type incompatibility probability to go up.

The flow tissue-type incompatibility has been increasing in the US since 2009 (see Table 1). For this reason, we use the 2017/2018 value $\theta = 0.068$ in our benchmark analysis. To assess the effects of changing θ, we conduct a sensitivity analysis for our model's numerical predictions when θ changes.

There is one unforeseen effect of changing θ in our model for the other underlying parameters. As we do not directly observe the living-donor pairing probabilities (α_X), we have to calibrate these using the actual direct living-donor transplant flows ($\boldsymbol{\lambda}_X$) and new patient arrival flows ($\boldsymbol{\pi}_X$) as well as the probabilities of finding a compatible paired donor ($p_X^{\mathbf{l}}$) as

$$\alpha_X = \frac{\boldsymbol{\lambda}_X}{p_X^{\mathbf{l}} \boldsymbol{\pi}_X},$$

where $p_X^{\mathbf{l}}$ is a decreasing linear function of θ (see Section 2.3). Thus, each α_X is an increasing function of θ. Hence, as θ goes up, we expect more paired patients to arrive with living donors to match the data.

Therefore, we conduct two sets of stress tests: We first fix each α_X at the benchmark level reported in Table 6 and adjust $\boldsymbol{\lambda}_X$ as we change θ according to the formula above. Then we allow changes in θ to affect each α_X by keeping $\boldsymbol{\lambda}_X$ at its benchmark level. We report the results of these tests in Figure 3 and Figure 4 in Appendix E.

When only θ changes at fixed (α_X) with an average equal to 0.288, as ρ increases the importance of varying θ disappears. The service rate and living-donor transplant number changes are shown in Figure 3 for $\theta \in \{0.047, 0.068, 0.089, 0.11, 0.131\}$. The number of annual direct living-donor transplants ranges from 5091 to 4644 with increasing θ. The marginal gain through regular exchange ranges in $983-1593$ additional transplants per year, and this gain is increasing in θ. Each additional 10 percent participation in incentivized exchange adds an additional $184-168$ transplants annually, and the increase is inversely proportional to θ.[24]

We also report how much the assumptions we made affect our model's predictions. Our theo-

[23] Different sources report higher θ values in the earlier literature (for example see Zenios, Woodle, and Ross, 2001 who cite θ as 0.11).

[24] It is important to emphasize that a change in θ mostly affects the number of patients benefitting from regular exchange. Its impact on the benefit of incentivized exchange is much smaller.

retical model makes two important assumptions. First, it models the flow of patients and donors as a continuum process, which effectively means that the markets are large. This minimizes the role of tissue-type incompatibility in the model. Second, it assumes that tissue-type incompatibility is the same for each patient, while we know that patients are heterogenous in their sensitization levels against tissue types of donors in real life. We inspect the implications of these two assumptions by contrasting the numerical predictions of the model with the simulations in Appendix D. The simulation model considers a finite market about one twentieth the size of the whole US (roughly the size of a small transplant region with about 1750 annual new patient arrivals) and inspects the outcomes after 15 years of running. It assumes the tissue-type incompatibility probabilities of patients are distributed according to the US patient flow statistics from OPTN. Simulations show that these two main simplifying assumptions only minimally impact the predicted number of additional patients benefitting from incentivized exchange, though they do impose costs on the prediction of the overall number of patients benefitting from regular exchange.

For example, Table 4 (in Column 5) shows that our theoretical model predicts 2 percent of additional paired patients will receive living donor transplantation for every $\Delta\rho = 10$ percent increase in participation of compatible pairs in incentivized exchange. Table 12 in Appendix D shows that this percentage gain is 1.9 or 2.1 (in Column 5) in the simulations (depending on whether compatible $X-X$ type pairs are incentivized or not). Although our model and simulations predict very similar benefits from incentivized exchange, our theoretical model predicts that 12.6 percent of pairs will benefit from regular exchange, while the simulation model only predicts 11 percent.

The main rationale for this difference is that, whenever a blood-type compatible pair is not tissue-type compatible, the patient of the pair is likely to have a higher tissue-type incompatibility probability than the average patient. These pairs constitute most of the incompatible overdemanded pairs that are utilized under regular kidney exchange. In contrast, whenever a blood-type compatible pair is also tissue-type compatible, the patient of the pair is likely to have a lower tissue-type incompatibility probability than the average patient. This observation reduces the role of tissue-type incompatibility in incentivized exchange compared to its role in regular kidney exchange. As a result, our model does a good job measuring the numbers of patients benefitting from incentivized exchange in finite markets. However, its precision is slightly inferior for measuring patients benefitting from regular exchange.[25]

We report additional stress test results regarding the effect of the availability of three-way exchange technology in Appendix D.

6 Conclusion

As the need for transplant kidneys is at an all time high, the efforts to increase living donation continue. For example, the Living Donor Protection Act of 2017 introduced in the US Congress is aimed at removing some of the disincentives to living donation by ensuring job protections for organ

[25]Therefore, the implications of our large market assumptions are even more benign in this paper for policy purposes than Roth, Sönmez, and Ünver (2007) who ignored tissue-type incompatibility and primarily focused on regular exchange as the policy question.

donors who need to take medical leave to recover from organ donation, and insurance protections so organ donors are not denied or charged higher premiums because they donated an organ.[26] Similarly the National Kidney Foundation issued a statement in January 2019 calling on Congress and the US Administration to make organ transplantation a top priority, identifying living donation as a critical area to be addressed from a legislative and regulatory standpoint.[27] Most recently in July 2019, President Trump signed the *kidney care executive order*, which aims to remove financial barriers to living organ donation.[28] Our proposal for incentivized exchange is in line with these recent efforts.

In practice, although our incentivization scheme can be made available to any compatible pair participating in exchange, prioritizing patients from these pairs for a possible repeat transplant requires the consent of the authorities that manage the deceased-donor queue. In the US, deceased-donor allocation is managed by OPTN and its subsidiary UNOS, which also operates a nationwide living-donor kidney exchange program. Therefore, one possibility is using our proposed incentive scheme only for compatible pairs who participate in the UNOS kidney exchange program. Providing the incentives only to participants of the UNOS kidney exchange program may also lead to consolidation, mitigating the welfare loss from a fragmanted kidney exchange market. In a recent paper, Agarwal et al. (2018) observe that the US kidney exchange programs are fragmented and as a result there is a significant transplant loss from (1) having multiple small programs running in parallel instead of a unified large one and (2) some small hospitals not being able to participate in any program due to prohibitive administrative costs. By encouraging participation of compatible pairs, the UNOS program can become a focal program that attracts the largest number of pairs, partially mitigating this welfare loss.[29]

While restricting the incentives only to the participants of the UNOS kidney exchange system is one way to implement incentivized exchange, it is not the only way. This is important because, restricting the incentives only to participants of the UNOS kidney exchange program may face some opposition. An alternative implementation could provide the incentives to any compatible pair regardless of which kidney exchange program they join. Any such pair provides valuable benefits to the entire pool of patients by reducing the demand for deceased-donor kidneys, and thus providing incentives to them is in the spirit of providing similar incentives for prior living donors.

Two key aspects of our proposal are inclusion of compatible pairs in exchange (to better utilize living donors) and an adjusted priority ranking in the deceased-donor queue (to incentivize them to participate in exchange). Incentivized exchange is related to three sparsely practiced variants of kidney exchange. In conclusion, we compare and contrast incentivized exchange with these variants.

An *altruistically unbalanced exchange* involves a kidney exchange between one compatible and

[26] See `https://www.congress.gov/bill/115th-congress/house-bill/1270` (retrieved on 01/28/2019).

[27] See `https://www.kidney.org/news/nkf-statement-path-forward-increasing-kidney-transplantation` (retrieved on 01/28/2019).

[28] See `https://www.whitehouse.gov/presidential-actions/executive-order-advancing-american-kidney-health/` (retrieved on 09/20/2019).

[29] See an earlier draft of the current paper, Sönmez and Ünver (2015), for a formal analysis of such a participation game.

one incompatible pair. Ross and Woodle (2000) dismisses these exchanges on ethical grounds. Their concern is potential coercion of compatible pairs who have nothing to gain from exchange. In contrast, exchange is no longer "altruistically unbalanced" under incentivized exchange, since patients of participating pairs are insured against a repeat kidney failure.

Under an *indirect exchange*, the donor of an incompatible pair donates a kidney to the deceased-donor queue in exchange for increased priority for his patient in the deceased-donor queue. Hence, this variant involves an exchange between an incompatible pair and the deceased-donor queue. Ross and Woodle (2000) object to indirect exchange for blood-type-incompatible pairs, but support it for blood-type-compatible (but tissue-type-incompatible) pairs. Consider a blood-type O patient with a blood-type A donor. Under an indirect exchange, the pair donates a blood-type A kidney to the donor queue in exchange for priority in the blood-type O deceased-donor queue. That is, they receive priority for a more highly sought-after blood-type kidney than the kidney they donate. This is the basis of the Ross and Woodle (2000) objection:

> "The indirect ABO-incompatible exchange does create a new ethical concern because it may increase the vulnerability of O blood group recipients. If mechanisms can be developed to avoid increasing the waiting time for blood group O recipients, we would support the implementation of the indirect ABO-incompatible exchange."

In contrast, they support indirect exchange for blood-type-compatible pairs because those pairs either donate the same blood-type or a more highly sought-after blood-type kidney than the one they receive priority for. While incentivized exchange is also based on priority in the deceased-donor queue, there are two key differences: First, an incentivized pair donates a kidney of a more highly sought-after blood type than its patient receives priority for. Indeed, incentivized kidney mainly benefits the blood-type O patient population. Second, unlike indirect exchange, the priority is only used if the patient needs a repeat transplant. Both factors address Ross and Woodle's ethical considerations.

A *voucher for a chronologically incompatible* pair (Veale et al., 2017) involves priority for a (typically young) patient of a pair for a future transplant in exchange for a donation from an older donor today. The donor will be too old to donate when the patient is expected to need a transplant. Observe that this variant is very similar to indirect exchange, and indeed it can be interpreted as an intertemporal version of indirect exchange. Therefore, the same ethical considerations from Ross and Woodle (2000) apply. That is, the case for these exchanges is stronger when the pair is blood-type compatible than when they are blood-type incompatible. Unlike an incentivized exchange or an indirect exchange, the first three of these intertemporal exchanges were organized by the National Kidney Registry, which arranges kidney chains initiated by good-samaritan donors.[30] The older donor starts a chain today, and the younger patient receives priority for a kidney at the end of a chain when he needs a transplant in the future. However, these chains almost never end with a blood-type O kidney, and indeed they are likely to end with a blood-type AB kidney. Hence, honoring the voucher may require artificially terminating a kidney chain, especially if the patient is of blood type O. Perhaps motivated by these concerns, Veale et al. (2017) suggest that patients

[30]These chains are introduced by Roth et al. (2006) and the proof of concept is documented in Rees et al. (2009).

also be prioritized in the deceased-donor queue in case the patient cannot be placed at the end of a kidney chain. Conceptually, incentivized exchange is similar, but it avoids the above-mentioned shortcomings since incentivized pairs are blood-type compatible. In summary, incentivized exchange harbors all the positive elements of the above variants of kidney exchange without suffering from their shortcomings.

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Appendix A Main Analytical Results

In this appendix, we provide the omitted proof of Theorem 1, derivation of waiting times, and additional comparative results. To do that, we first provide a formal model of tissue-type incompatibility: Each patient has a type, depending on against which subset of HLA tissue proteins he has preformed antibodies. We study the limit as the number of types goes to infinity. First, fix the number of types to a finite k. The probability that a patient is of type i is $m_{i,k} > 0$, so that $\sum_i m_{i,k} = 1$. Let $\theta_{i,k}$ be the tissue-type incompatibility probability between any donor and patient of type i. If a donor is tissue-type compatible with a type i patient, then the donor is tissue-type compatible with all patients of type i. We take the number of types, k, to infinity and make some regularity assumptions on the growth of $m_{i,k}$ and $\theta_{i,k}$ in the limit. See Appendix F for details. These assumptions hold for the special case when $\theta_{i,k} = \theta < 1$ and $m_{i,k} \to 0$ for every patient type i as $k \to \infty$ (Lemma 7). In the current appendix, as well as for the results in the main text, we use this special case.

We also define steady states formally: A **state** is defined through the measures of type $X - Y$ pairs who have waited t years in the patient pool, denoted by $(X-Y, t)$, and blood-type X unpaired patients who have waited t years in the patient pool, denoted by (X, t), for all blood types X and Y and waiting time t. We say that the population under a given policy of transplantation is at a **steady state** when the measures of all $(X - Y, t)$ and (X, t) are constant through time, i.e., the state does not change over time.

A.1 Optimal Regular and Incentivized Exchange

We first formally categorize pair types in the following three classes. The naming of these classes is based on the comparison of the flows of the type and its reciprocal type for ABO-i optimal exchange (Assumptions 1 and 2). The types $X-Y$ with a weakly lower flow than that of $Y-X$ are overdemanded, while the ones with the weakly higher flow are underdemanded. Remaining types are referred to as self-demanded, as each such pair is matched with another pair of the same type in ABO-i optimal exchange:

- **Overdemanded Types**: These are pair types $X - Y$ such that $Y \triangleright X$ and $Y \neq X$ and pair type $A - B$. There are six of these types, $A - O$, $A - B$, $B - O$, $AB - O$, $AB - A$ and $AB - B$.
- **Self-demanded Types**: These are pair types $X - X$. There are four of these types, $O - O$, $A - A$, $B - B$, and $AB - AB$.
- **Underdemanded Types**: These are pair types $X - Y$ such that $X \triangleright Y$ and $X \neq Y$ and pair type $B - A$. There are six of these types, $O - A$, $O - B$, $O - AB$, $A - AB$, $B - A$, and $B - AB$.

The following lemma characterizes feasible exchanges. It is useful in the proof of Theorem 1. Similar results also appear in Roth, Sönmez, and Ünver (2007) and Ünver (2010), so we skip its proof.

Lemma 1 (Exchange blood-type feasibility) *An underdemanded-type pair can be matched only with an overdemanded-type pair in an exchange. An overdemanded-type pair can be matched*

with pairs with types from each of the three categories. A self-demanded-type pair can be matched with a same-type or overdemanded-type pair. In particular, the following results hold:

- *An underdemanded-type $O-A$ (or $O-B$) pair can be matched only with a pair from overdemanded types $A-O$ (or $B-O$) or $AB-O$. An underdemanded-type $A-AB$ (or $B-AB$) pair can be matched only with a pair from overdemanded types $AB-A$ (or $AB-B$) or $AB-O$. An underdemanded-type $O-AB$ pair can be matched only with an overdemanded-type $AB-O$ pair.*
- *An overdemanded-type $A-B$ (or underdemanded-type $B-A$) pair can be matched only with a pair from its reciprocal type $B-A$ (or $A-B$); or from overdemanded types $B-O$ (or $A-O$), $AB-A$ (or $AB-B$), or $AB-O$.*
- *A self-demanded-type $X-X$ pair can be matched with a same-type pair. Additionally, a type $O-O$ pair can be matched only with a pair from overdemanded types $A-O$, $B-O$, or $AB-O$; a type $A-A$ (or $B-B$) pair can be matched only with a pair from overdemanded types $AB-A$ (or $AB-B$) or $AB-O$; and a type $AB-AB$ pair can be matched only with a pair from overdemanded types $AB-A$, $AB-B$, or $AB-O$.*

Proof of Theorem 1. Suppose Assumptions 1 and 2 hold. Under the proposed policy, by Lemma 6 in Appendix F, all self-demanded-type pairs can be matched with their own-type pairs as soon as they arrive.

Similarly, type $A-B$ pairs, which have a weakly lower flow rate than that of type $B-A$ by Assumption 2, can be matched as soon as they arrive with type $B-A$ pairs (Lemma 4 in Appendix F). Hence, under this policy some type $B-A$ pairs will remain in the exchange pool. These pairs can only be matched with some overdemanded-type pairs by Lemma 1, as type $A-B$ pairs are already committed to other type $B-A$ pairs.

Next consider underdemanded-type pairs except those of $B-A$. These are type $Y-X$ pairs such that $Y \neq X$ and $Y \triangleright X$. By Assumption 1, we have $\theta p_Y \alpha_X \boldsymbol{\pi}_X \leq p_X \alpha_Y \boldsymbol{\pi}_Y$. By Lemma 1, they can only be matched with overdemanded-type pairs. Recall that the flow of each type $Y-X$ pair to the exchange pool is $p_X \alpha_Y \boldsymbol{\pi}_Y$. Their reciprocal type $X-Y$, which is overdemanded, has flow $\theta p_Y \alpha_X \boldsymbol{\pi}_X \leq p_X \alpha_Y \boldsymbol{\pi}_Y$. Hence, we can match all such overdemanded-type $X-Y$ pairs as soon as they enter the pool with their reciprocal-type pairs (Lemma 5 in Appendix F). As all overdemanded- and self-demanded-type pairs are matched as soon as they arrive, by Lemma 1, the proposed policy achieves the maximum measure of pairs matched. At steady state, as no incompatible overdemanded-type or self-demanded-type pair waits in the pool (and moreover, get immediately matched and help one additional pair), the maximum mass of possible exchanges is also conducted in this manner in any closed time interval.

On the other hand, if we do not conduct the exchanges immediately whenever they become available but only after some time interval, then some of the patients will not survive. Hence, when we do not conduct the exchanges as soon as possible, we will match a strictly smaller mass of pairs than we would have matched under the proposed policy. ■

A.2 Finding Waiting Times for Transplantation

In this subsection of this appendix, we explain how we find waiting times for deceased-donor and living-donor transplants at the steady state using our dynamic continuum model.

A.2.1 Only Deceased-Donor Transplantation

We start when the only available transplantation regime is deceased donation. In this case, at any time the longest-waiting cohort of blood-type X patients receive the incoming blood-type X deceased-donor kidneys. Let this cohort have arrived $t_X^{\mathbf{d}}$ years before the current time. Assuming deceased-donor kidneys are the only source of transplants, at steady state we have

$$[\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X]S(t_X^{\mathbf{d}}) = \boldsymbol{\delta}_X.$$

Hence, the time spent on the blood-type X deceased-donor queue at steady state, or equivalently the transplant waiting time for blood-type X patients, can be found as

$$t_X^{\mathbf{d}} = S^{-1}\left(\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X}\right) = S^{-1}\left(s_X^{\mathbf{d},dec}\right).$$

A.2.2 Deceased-Donor/Direct Living-Donor Transplantation

When additionally direct living-donor transplantation is available, at any time the longest-waiting cohort of blood-type X patients without compatible donors receive the incoming blood-type X deceased-donor kidneys. Let this cohort have arrived $t_X^{\mathbf{l}}$ years before the current time. At steady state, we have $\boldsymbol{\pi}_X^{\mathbf{l}} S(t_X^{\mathbf{l}}) = \boldsymbol{\delta}_X$, and therefore the time spent on the blood-type X deceased-donor queue by the receiving cohort can be found as

$$t_X^{\mathbf{l}} = S^{-1}\left(\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X^{\mathbf{l}}}\right) = S^{-1}\left(\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X - (1-\phi^{\mathbf{l}})\boldsymbol{\lambda}_X}\right) = S^{-1}\left(s_X^{\mathbf{l},dec}\right).$$

All living-donor transplants are carried out instantaneously; thus, their waiting time is zero.

A.2.3 Adding Regular or Incentivized Exchange

Next, we derive waiting times for transplantations when regular or incentivized exchange is also feasible in addition to deceased-donor and direct living-donor transplantation.

Recall that for all incentivized-exchange-eligible pairs, i.e., of all types $X - Y$ such that $Y \triangleright X$, $Y \neq X$, and the patient and donor have no tissue-type incompatibility, $\rho_{X-Y} \in [0,1]$ is the fraction that participate in incentivized exchange. Let $\rho = (\rho_{X-Y})_{Y \triangleright X, Y \neq X}$ be the vector of such fractions. We use the terms "regular exchange" and "incentivized exchange with $\rho = 0\%$" interchangeably. To determine the steady-state outcomes, we introduce certain flow rates.[31] For each blood type X and each $Y \neq X$, let

$$\boldsymbol{\pi}_{X-Y} = \begin{cases} [\theta + \rho_{X-Y}(1-\theta)]p_Y\alpha_X\boldsymbol{\pi}_X & \text{if } Y \triangleright X \\ p_Y\alpha_X\boldsymbol{\pi}_X & \text{otherwise} \end{cases} \tag{1}$$

refer to the pair-type $X - Y$ flow to the exchange pool. Let the **incentivized pair flow** relevant

[31] Some of these were previously defined throughout Section 4.

for blood type X be given by

$$\boldsymbol{\kappa}_X \;= \Big(\sum_{Y\,:\,Y \triangleright X \;\&\; Y \neq X} \rho_{X-Y}(1-\theta) p_Y \Big) \alpha_X \boldsymbol{\pi}_X. \tag{2}$$

Observe that $\phi^{\mathbf{l}}\boldsymbol{\kappa}_X$ is the reentry flow of previously incentivized blood-type X patients to the patient pool. These patients will be prioritized in the deceased-donor queue of blood type X and will not wait upon reentry. Thus, the effective flow rate of deceased-donor kidneys for nonprioritized blood-type X patients is $\boldsymbol{\delta}_X - \phi^{\mathbf{l}}\boldsymbol{\kappa}_X$. We also have

$$\boldsymbol{\pi}_X^{np\&u} = \underbrace{(1-\alpha_X)\boldsymbol{\pi}_X}_{\text{new unpaired}} + \underbrace{\phi^{\mathbf{d}}\boldsymbol{\delta}_X}_{\text{reentry / deceased}} + \underbrace{\phi^{\mathbf{l}}[\boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X + \boldsymbol{\iota}_X - \boldsymbol{\kappa}_X]}_{\text{reentry / all live minus incentivized}} \tag{3}$$

as the total **nonprioritized and unpaired blood-type X patient flow**.

We define the following ratios:

1. The ratio of the deceased-donor effective flow for nonprioritized patients to the nonprioritized and unpaired patient flow is
$$r_X = \frac{\boldsymbol{\delta}_X - \phi^{\mathbf{l}}\boldsymbol{\kappa}_X}{\boldsymbol{\pi}_X^{np\&u}}. \tag{4}$$
2. For each underdemanded type $X-Y$ except $B-A$ (i.e., $X \neq Y$ and $X \triangleright Y$), the ratio of the flow of incompatible or incentivized type $Y-X$ pairs to the total flow of type $X-Y$ pairs is
$$r_{X-Y} = \frac{\boldsymbol{\pi}_{Y-X}}{\boldsymbol{\pi}_{X-Y}} = \frac{[\theta + \rho_{Y-X}(1-\theta)] p_X \alpha_Y \boldsymbol{\pi}_Y}{p_Y \alpha_X \boldsymbol{\pi}_X}.$$
3. For the remaining underdemanded type $B-A$, the ratio of type $A-B$ flow to type $B-A$ flow is
$$r_{B-A} = \frac{\boldsymbol{\pi}_{A-B}}{\boldsymbol{\pi}_{B-A}} = \frac{p_B \alpha_A \boldsymbol{\pi}_A}{p_A \alpha_B \boldsymbol{\pi}_B}.$$

The first ratio, r_X, is less than one because of our assumption that there is a shortage of deceased-donor kidneys for unpaired new entrants, i.e., $(1-\alpha_X)\boldsymbol{\pi}_X > \boldsymbol{\delta}_X$. The second ratio, r_{X-Y}, is less than one by Assumption 3. Finally, the last ratio, r_{B-A}, is less than or equal to one by Assumption 2. Ratio r_X would be a service rate if we wanted to allocate all blood-type X deceased donors that are reserved for nonprioritized patients to nonprioritized and unpaired blood-type X patients. For an underdemanded type $X-Y$, ratio r_{X-Y} would be a service rate for living-donor transplants if type $X-Y$ pairs did not receive any deceased-donor transplants but only participated in exchange. In these cases, the waiting time for a deceased-donor transplant for nonprioritized and unpaired blood-type X patients would be

$$t_X = S^{-1}\Big(\frac{\boldsymbol{\delta}_X - \phi^{\mathbf{l}}\boldsymbol{\kappa}_X}{\boldsymbol{\pi}_X^{np\&u}}\Big),$$

and the waiting time of underdemanded-type $X-Y$ pairs would be

$$t_{X-Y} = S^{-1}\Big(\frac{\boldsymbol{\pi}_{Y-X}}{\boldsymbol{\pi}_{X-Y}}\Big).$$

However, underdemanded-type pairs have another option besides exchange. If deceased donors become available earlier than the exchange option, they will receive deceased-donor transplants. As we mentioned in the main text, we assume that patients accept the first donor who is offered to

them, either through deceased-donor allocation or exchange. Hence, the patient of a type $X-Y$ pair will never wait for a type $Y-X$ pair for exchange if a deceased donor becomes available first, i.e., if $t_{X-Y} > t_X$. As waiting times are strictly decreasing functions of the r ratios defined above, we need to compare these ratios in an iterative manner to decide whether pairs of one or more underdemanded types will also receive deceased-donor transplants.

To this end, we first define $X-Y_1,\ldots,X-Y_{k(X)}$ as the ordered list of underdemanded types according to ascending r_{X-Y} ratios, where we have $k(O)=3$, $k(B)=2$, $k(A)=1$, and $k(AB)=0$ as the respective numbers of underdemanded pair types whose patients have blood types O, B, A, and AB. We define the following potential pooling ratio for each $\ell=0,\ldots,k(X)$:

$$r_{X,X-Y_1,\ldots,X-Y_\ell} = \frac{\boldsymbol{\delta}_X - \phi^{\mathbf{1}}\boldsymbol{\kappa}_X + \boldsymbol{\pi}_{Y_1-X} + \ldots + \boldsymbol{\pi}_{Y_\ell-X}}{\boldsymbol{\pi}_X^{np\&u} + \boldsymbol{\pi}_{X-Y_1} + \ldots + \boldsymbol{\pi}_{X-Y_\ell}}. \tag{5}$$

Exchange regime iterative pooling procedure for unpaired and paired patients:

Fix a blood type X. We iteratively consider the following procedure starting with $\ell=0$.

Step ℓ: Suppose types $X-Y_1,\ldots,X-Y_\ell$ have already been deemed to be receiving both deceased-donor and exchange transplants.

- If $r_{X-Y_{\ell+1}} < r_{X,X-Y_1,\ldots,X-Y_\ell}$ then type $X-Y_{\ell+1}$ pairs receive exchange and deceased-donor transplants at the same time together with the nonprioritized and unpaired blood-type X patients and type $X-Y_1,\ldots,X-Y_\ell$ pairs. We continue with Step $\ell+1$.
- If $r_{X-Y_{\ell+1}} \geq r_{X,X-Y_1,\ldots,X-Y_\ell}$ then all type $X-Y_{\ell+1},\ldots,X-Y_{k(X)}$ pairs only receive exchange transplants but no transplants from deceased donors. We terminate the procedure.

Based on this procedure, we state the following theorem:

Theorem 2 (Waiting times under regular and incentivized exchange) *Suppose Assumptions 2 and 3 hold. Consider the ABO-i deceased-donor allocation and incentivized-exchange policies with a given incentivized-exchange participation-rate vector $\rho=(\rho_{X-Y})_{Y\triangleright X, Y\neq X}$ (which can possibly be zero). Consider a blood type X. Then the following statements hold:*

1. *Blood-type X patients, who are in overdemanded-type or self-demanded-type pairs and who have either incompatible donors or are eligible and willing to participate in incentivized exchange, participate in exchange immediately upon their arrival to the patient pool.*
2. *Suppose the patients in pairs of underdemanded types $X-Y_1,\ldots,X-Y_{\ell(X)}$ receive exchange and deceased-donor transplants, while the patients in pairs of underdemanded types $X-Y_{\ell(X)+1},\ldots,X-Y_{k(X)}$ receive only exchange transplants for some $\ell(X)\in\{0,\ldots,k(X)\}$.*[32] *Then*
   - *nonprioritized and unpaired blood-type X patients and the patients of type $X-Y_1,\ldots,X-Y_{\ell(X)}$*

[32] $\ell(X)=0$ refers to the case where no underdemanded type with blood-type X patient receives deceased-donor transplant, and $\ell(X)=k(X)$ refers to the case where all underdemanded types with blood-type X patients receive both exchange and deceased-donor transplants.

pairs wait for a deceased-donor (or exchange) transplant for the duration

$$t_X^{\mathbf{i}} \;=\; S^{-1}\left(\frac{\boldsymbol{\delta}_X - \phi^{\mathbf{l}}\boldsymbol{\kappa}_X + \boldsymbol{\pi}_{Y_1-X} + \ldots + \boldsymbol{\pi}_{Y_{\ell(X)}-X}}{\boldsymbol{\pi}_X^{np\&u} + \boldsymbol{\pi}_{X-Y_1} + \ldots + \boldsymbol{\pi}_{X-Y_{\ell(X)}}}\right), \text{ and,} \tag{6}$$

- *for all $\ell \in \{\ell(X)+1, \ldots, k(X)\}$, type $X - Y_\ell$ pairs are exclusively matched through exchange and wait for an exchange transplant for the duration*

$$t_{X-Y_\ell}^{\mathbf{i}} \;=\; S^{-1}\left(\frac{\boldsymbol{\pi}_{Y_\ell-X}}{\boldsymbol{\pi}_{X-Y_\ell}}\right) \leq t_X^{\mathbf{i}}. \tag{7}$$

The average waiting time to a transplant for all blood-type X patients is

$$t_X^{\mathbf{i},all} \;=\; \frac{\left[\boldsymbol{\delta}_X - \phi^{\mathbf{l}}\boldsymbol{\kappa}_X + \sum_{\ell=1}^{\ell(X)} \boldsymbol{\pi}_{Y_\ell-X}\right] t_X^{\mathbf{i}} + \sum_{\ell=\ell(X)+1}^{k(X)} \left[\boldsymbol{\pi}_{Y_\ell-X}\; t_{X-Y_\ell}^{\mathbf{i}}\right]}{\boldsymbol{\delta}_X + \boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X + \boldsymbol{\iota}_X} \tag{8}$$

Proof. The proof follows from the procedure discussed before the statement of the theorem. Since blood-type X patients with compatible paired donors and blood-type X patients with incompatible but blood-type-compatible donors have zero waiting time, Equation 8 is established. ■

When $\rho = 0$, we will refer to $t_X^{\mathbf{i}}$ as $t_X^{\mathbf{e}}$ and $t_X^{\mathbf{i},all}$ as $t_X^{\mathbf{e},all}$.

A.3 Welfare Consequences of Transplant Regimes on Access of Patients to Living-Donor Transplantation

We next present a result, which formulates how access to living-donor transplantation differs across blood types with the introduction of each transplantation modality. For this analytical result, we consider a baseline scenario where no blood type has an advantage over another for access to transplantation beyond the asymmetry induced by blood-type compatibility and the impact of the transplantation modalities analyzed. We present a corresponding result for access to deceased-donor transplantation, formulated through waiting times, in the last subsection of this appendix.

Theorem 3 *Suppose Assumptions 2 and 3 hold. Let $p_A > p_B$. Suppose that $\alpha_X = \alpha$ for any blood type X, and $\frac{\boldsymbol{\pi}_X}{p_X} = \frac{\boldsymbol{\pi}_Y}{p_Y}$ for any two blood types X and Y. Suppose also that the fraction of pairs taking the incentivized-exchange option is uniform at a fixed $\rho < 1$ for any eligible type. Then:*

1. *For direct living-donor transplantation, the access to living donation is ranked as*

$$\frac{\boldsymbol{\lambda}_O}{\boldsymbol{\pi}_O} < \frac{\boldsymbol{\lambda}_B}{\boldsymbol{\pi}_B} < \frac{\boldsymbol{\lambda}_A}{\boldsymbol{\pi}_A} < \frac{\boldsymbol{\lambda}_{AB}}{\boldsymbol{\pi}_{AB}}.$$

2. *Kidney exchange, in addition to direct living-donor transplantation, by itself increases access to living-donor transplantation for patients of blood type B the most, patients of blood type A next, and patients of blood types AB and O equally and last: $\frac{\boldsymbol{\epsilon}_B}{\boldsymbol{\pi}_B} > \frac{\boldsymbol{\epsilon}_A}{\boldsymbol{\pi}_A} > \frac{\boldsymbol{\epsilon}_{AB}}{\boldsymbol{\pi}_{AB}} = \frac{\boldsymbol{\epsilon}_O}{\boldsymbol{\pi}_O}$. With the inclusion of kidney exchange, overall access to living donation is ranked as*

$$\frac{\boldsymbol{\lambda}_O + \boldsymbol{\epsilon}_O}{\boldsymbol{\pi}_O} < \frac{\boldsymbol{\lambda}_B + \boldsymbol{\epsilon}_B}{\boldsymbol{\pi}_B} = \frac{\boldsymbol{\lambda}_A + \boldsymbol{\epsilon}_A}{\boldsymbol{\pi}_A} < \frac{\boldsymbol{\lambda}_{AB} + \boldsymbol{\epsilon}_{AB}}{\boldsymbol{\pi}_{AB}} = \alpha.$$

3. *Incentivized exchange, in addition to regular exchange and direct living-donor transplantation, by itself increases access to living-donor transplantation for patients of blood type O the most, patients of blood types A and B equally and next, and does not increase access for patients of*

blood type AB: $\frac{\boldsymbol{\iota}_O}{\boldsymbol{\pi}_O} > \frac{\boldsymbol{\iota}_A}{\boldsymbol{\pi}_A} = \frac{\boldsymbol{\iota}_B}{\boldsymbol{\pi}_B} > \frac{\boldsymbol{\iota}_{AB}}{\boldsymbol{\pi}_{AB}} = 0$. *With the inclusion of either version of incentivized exchange, overall access to living donation is ranked as*

$$\frac{\boldsymbol{\lambda}_O + \boldsymbol{\epsilon}_O + \boldsymbol{\iota}_O}{\boldsymbol{\pi}_O} < \frac{\boldsymbol{\lambda}_B + \boldsymbol{\epsilon}_B + \boldsymbol{\iota}_B}{\boldsymbol{\pi}_B} = \frac{\boldsymbol{\lambda}_A + \boldsymbol{\epsilon}_A + \boldsymbol{\iota}_B}{\boldsymbol{\pi}_A} < \frac{\boldsymbol{\lambda}_{AB} + \boldsymbol{\epsilon}_{AB} + \boldsymbol{\iota}_{AB}}{\boldsymbol{\pi}_{AB}} = \alpha.$$

Proof of Theorem 3. Let $p_A > p_B$. Suppose that $\alpha_X = \alpha$ for any blood type X, and $\frac{\boldsymbol{\pi}_X}{\boldsymbol{\pi}_Y} = \frac{p_X}{p_Y}$ for any two blood types X and Y. Also assume that the fraction of pairs taking the incentivized-exchange option is uniform at a fixed $\rho < 1$ for any eligible type.

1. We consider $\boldsymbol{\lambda}_X$, the overall flows of pairs with blood-type X patients participating in direct living-donor-transplantation regime for each blood type X:

$$\begin{aligned}
\frac{\boldsymbol{\lambda}_O}{\boldsymbol{\pi}_O} &= \frac{(1-\theta)p_O\alpha\boldsymbol{\pi}_O}{\boldsymbol{\pi}_O} = (1-\theta)p_O\alpha,\\
\frac{\boldsymbol{\lambda}_A}{\boldsymbol{\pi}_A} &= \frac{(1-\theta)(p_O+p_A)\alpha\boldsymbol{\pi}_A}{\boldsymbol{\pi}_A} = (1-\theta)(p_O+p_A)\alpha,\\
\frac{\boldsymbol{\lambda}_B}{\boldsymbol{\pi}_B} &= \frac{(1-\theta)(p_O+p_B)\alpha\boldsymbol{\pi}_B}{\boldsymbol{\pi}_B} = (1-\theta)(p_O+p_B)\alpha, \text{ and}\\
\frac{\boldsymbol{\lambda}_{AB}}{\boldsymbol{\pi}_{AB}} &= \frac{(1-\theta)\alpha\boldsymbol{\pi}_{AB}}{\boldsymbol{\pi}_{AB}} = (1-\theta)\alpha.
\end{aligned}$$

Thus,

$$\frac{\boldsymbol{\lambda}_O}{\boldsymbol{\pi}_O} < \frac{\boldsymbol{\lambda}_A}{\boldsymbol{\pi}_A}, \frac{\boldsymbol{\lambda}_B}{\boldsymbol{\pi}_B} < \frac{\boldsymbol{\lambda}_{AB}}{\boldsymbol{\pi}_{AB}}.$$

Moreover, since $p_B < p_A$, we have $\frac{\boldsymbol{\lambda}_B}{\boldsymbol{\pi}_B} < \frac{\boldsymbol{\lambda}_A}{\boldsymbol{\pi}_A}$.

2. We consider $\boldsymbol{\epsilon}_X$, the overall flows of pairs that have blood-type X patients and participate in regular exchange, for each X:

$$\begin{aligned}
\frac{\boldsymbol{\epsilon}_O}{\boldsymbol{\pi}_O} &= \frac{\theta p_O\alpha(\boldsymbol{\pi}_O+\boldsymbol{\pi}_A+\boldsymbol{\pi}_B+\boldsymbol{\pi}_{AB})}{\boldsymbol{\pi}_O} = (\theta p_O + \theta p_A + \theta p_B + \theta p_{AB})\,\alpha = \theta\alpha,\\
\frac{\boldsymbol{\epsilon}_A}{\boldsymbol{\pi}_A} &= \frac{\theta p_O\alpha\boldsymbol{\pi}_A + \theta p_A\alpha\boldsymbol{\pi}_A + p_B\alpha\boldsymbol{\pi}_A + \theta p_A\alpha\boldsymbol{\pi}_{AB}}{\boldsymbol{\pi}_A} = (\theta p_O + \theta p_A + p_B + \theta p_{AB})\,\alpha,\\
\frac{\boldsymbol{\epsilon}_B}{\boldsymbol{\pi}_B} &= \frac{\theta p_O\alpha\boldsymbol{\pi}_B + p_B\alpha\boldsymbol{\pi}_A + \theta p_B\alpha\boldsymbol{\pi}_B + \theta p_B\alpha\boldsymbol{\pi}_{AB}}{\boldsymbol{\pi}_B} = (\theta p_O + p_A + \theta p_B + \theta p_{AB})\alpha, \text{ and}\\
\frac{\boldsymbol{\epsilon}_{AB}}{\boldsymbol{\pi}_{AB}} &= \frac{\theta p_O\alpha\boldsymbol{\pi}_{AB} + \theta p_A\alpha\boldsymbol{\pi}_{AB} + \theta p_B\alpha\boldsymbol{\pi}_{AB} + \theta p_{AB}\alpha\boldsymbol{\pi}_{AB}}{\boldsymbol{\pi}_{AB}} = (\theta p_O + \theta p_A + \theta p_B + \theta p_{AB})\,\alpha = \theta\alpha,
\end{aligned}$$

where the second equality in each line (except the last) follows from the assumption that $\frac{p_X}{\boldsymbol{\pi}_X}$ is constant among all X. Since $\theta < 1$ and $p_A, p_B > 0$, we have

$$\frac{\boldsymbol{\epsilon}_O}{\boldsymbol{\pi}_O} = \frac{\boldsymbol{\epsilon}_{AB}}{\boldsymbol{\pi}_{AB}} < \frac{\boldsymbol{\epsilon}_A}{\boldsymbol{\pi}_A}, \frac{\boldsymbol{\epsilon}_B}{\boldsymbol{\pi}_B}.$$

With the additional assumption $p_A > p_B$, we obtain $\frac{\boldsymbol{\epsilon}_A}{\boldsymbol{\pi}_A} < \frac{\boldsymbol{\epsilon}_B}{\boldsymbol{\pi}_B}$.

We consider each $\boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X$, the flow of direct living-donor and exchange transplants in total.

We have

$$\begin{aligned}
\frac{\boldsymbol{\lambda}_O + \boldsymbol{\epsilon}_O}{\boldsymbol{\pi}_O} &= (1-\theta)p_O\alpha + \theta(p_O + p_A + p_B + p_{AB})\alpha = (p_O + \theta p_A + \theta p_B + \theta p_{AB})\alpha, \\
\frac{\boldsymbol{\lambda}_A + \boldsymbol{\epsilon}_A}{\boldsymbol{\pi}_A} &= (1-\theta)(p_O + p_A)\alpha + (\theta p_O + \theta p_A + p_B + \theta p_{AB})\alpha = (p_O + p_A + p_B + \theta p_{AB})\alpha, \\
\frac{\boldsymbol{\lambda}_B + \boldsymbol{\epsilon}_B}{\boldsymbol{\pi}_B} &= (1-\theta)(p_O + p_B)\alpha + (\theta p_O + p_A + \theta p_B + \theta p_{AB})\alpha = (p_O + p_A + p_B + \theta p_{AB})\alpha, \text{ and} \\
\frac{\boldsymbol{\lambda}_{AB} + \boldsymbol{\epsilon}_{AB}}{\boldsymbol{\pi}_{AB}} &= (1-\theta)\alpha + \theta\alpha = \alpha.
\end{aligned}$$

Since $\theta < 1$ and $p_A, p_B, p_{AB} > 0$,

$$\frac{\boldsymbol{\lambda}_O + \boldsymbol{\epsilon}_O}{\boldsymbol{\pi}_O} < \frac{\boldsymbol{\lambda}_A + \boldsymbol{\epsilon}_A}{\boldsymbol{\pi}_A} = \frac{\boldsymbol{\lambda}_B + \boldsymbol{\epsilon}_B}{\boldsymbol{\pi}_B} < \frac{\boldsymbol{\lambda}_{AB} + \boldsymbol{\epsilon}_{AB}}{\boldsymbol{\pi}_{AB}} = \alpha.$$

3. Next we consider $\boldsymbol{\iota}_X$, the overall flow of pairs with blood-type X patients benefitting from incentivized exchange for each blood type X:

$$\begin{aligned}
\frac{\boldsymbol{\iota}_O}{\boldsymbol{\pi}_O} &= \frac{\rho(1-\theta)p_O\alpha\boldsymbol{\pi}_A + \rho(1-\theta)p_O\alpha\boldsymbol{\pi}_B + \rho(1-\theta)p_O\alpha\boldsymbol{\pi}_{AB}}{\boldsymbol{\pi}_O} = \rho(1-\theta)(p_A + p_B + p_{AB})\alpha, \\
\frac{\boldsymbol{\iota}_A}{\boldsymbol{\pi}_A} &= \frac{\rho(1-\theta)p_A\alpha\boldsymbol{\pi}_{AB}}{\boldsymbol{\pi}_A} = \rho(1-\theta)p_{AB}\alpha, \\
\frac{\boldsymbol{\iota}_B}{\boldsymbol{\pi}_B} &= \frac{\rho(1-\theta)p_B\alpha\boldsymbol{\pi}_{AB}}{\boldsymbol{\pi}_B} = \rho(1-\theta)p_{AB}\alpha, \text{ and} \\
\frac{\boldsymbol{\iota}_{AB}}{\boldsymbol{\pi}_{AB}} &= 0,
\end{aligned}$$

where the second equality in each line (except the last) follows from the assumption that $\frac{p_X}{\boldsymbol{\pi}_X}$ is constant among all X. Since $\rho > 0$, $\theta < 1$, and $p_A, p_B, p_{AB} > 0$,

$$0 = \frac{\boldsymbol{\iota}_{AB}}{\boldsymbol{\pi}_{AB}} < \frac{\boldsymbol{\iota}_A}{\boldsymbol{\pi}_A} = \frac{\boldsymbol{\iota}_B}{\boldsymbol{\pi}_B} < \frac{\boldsymbol{\iota}_O}{\boldsymbol{\pi}_O}.$$

We consider each $\boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X + \boldsymbol{\iota}_X$, i.e., direct living-donor, regular-exchange, and incentivized-exchange transplants in total. We have

$$\begin{aligned}
\frac{\boldsymbol{\lambda}_O + \boldsymbol{\epsilon}_O + \boldsymbol{\iota}_O}{\boldsymbol{\pi}_O} &= p_O\alpha + [\theta + \rho(1-\theta)](p_A + p_B + p_{AB})\alpha, \\
\frac{\boldsymbol{\lambda}_A + \boldsymbol{\epsilon}_A + \boldsymbol{\iota}_A}{\boldsymbol{\pi}_A} &= (p_O + p_A + p_B)\alpha + [\theta + \rho(1-\theta)]\, p_{AB}\alpha, \\
\frac{\boldsymbol{\lambda}_B + \boldsymbol{\epsilon}_B + \boldsymbol{\iota}_B}{\boldsymbol{\pi}_B} &= (p_O + p_A + p_B)\alpha + [\theta + \rho(1-\theta)]\, p_{AB}\alpha, \text{ and} \\
\frac{\boldsymbol{\lambda}_{AB} + \boldsymbol{\epsilon}_{AB} + \boldsymbol{\iota}_{AB}}{\boldsymbol{\pi}_{AB}} &= \alpha.
\end{aligned}$$

Since $\rho, p_{AB}, \alpha > 0$,

$$\frac{\boldsymbol{\lambda}_O + \boldsymbol{\epsilon}_O + \boldsymbol{\iota}_O}{\boldsymbol{\pi}_O} < \frac{\boldsymbol{\lambda}_A + \boldsymbol{\epsilon}_A + \boldsymbol{\iota}_A}{\boldsymbol{\pi}_A} = \frac{\boldsymbol{\lambda}_B + \boldsymbol{\epsilon}_B + \boldsymbol{\iota}_B}{\boldsymbol{\pi}_B} < \frac{\boldsymbol{\lambda}_{AB} + \boldsymbol{\epsilon}_{AB} + \boldsymbol{\iota}_{AB}}{\boldsymbol{\pi}_{AB}} = \alpha,$$

and they are all equal if and only if $\rho = 1$, because $\theta < 1$. ■

A.4 Consequences of Different Transplantation Regimes on Transplant Waiting Times

In this subsection, we state and prove a lemma that formalizes the marginal effects of living-donor exchange policies on the transplant waiting times of the following 29 groups of patients under some reasonable assumptions. These 29 groups are nonprioritized and unpaired patients of each blood type (4 groups), compatible pairs of overdemanded and self-demanded types (5 groups for overdemanded types and 4 groups for self-demanded types; recall that the overdemanded type $A-B$ pairs are never compatible), incompatible pairs of overdemanded and self-demanded types (6 groups for overdemanded types and 4 groups for self-demanded types), and pairs of underdemanded types (6 groups).

In addition to Assumptions 2 and 3, we also assume that the tissue-type incompatibility probability θ and the reentry rate of living-donor-transplant recipients $\phi^{\mathbf{l}}$ are sufficiently small. Formally, "for a vector of **sufficiently small** parameters x, some claim holds" means that "there exists some vector $\overline{x} \gg 0$ (i.e., all entries of the vector are larger than 0) such that for all x, $0 \leq x \leq \overline{x}$, that claim holds." These assumptions guarantee that all underdemanded-type pairs, except possibly type $B-A$, are pooled with their respective nonprioritized and unpaired patients for deceased-donor transplantation under the regular-exchange regime. Furthermore, we also assume that the difference between flows of pair types $B-A$ and $A-B$ is sufficiently small. This guarantees that $B-A$ pairs only participate in exchange and are never pooled for deceased-donor transplantation in all of exchange regimes we consider. This lemma will be used to prove Theorem 4, our last result of this appendix in the next subsection.

Lemma 2 (Consequences of regular and incentivized exchange) *Suppose Assumptions 2 and 3 hold for a given incentivized-exchange participation-fraction profile $\rho^* = (\rho^*_{X-Y})_{Y \triangleright X \,\&\, Y \neq X}$. Suppose also that reentry fraction of living-donor-transplant recipients $\phi^{\mathbf{l}}$, inflow difference between types $B-A$ and $A-B$ given as $p_A \alpha_B \boldsymbol{\pi}_B - p_B \alpha_A \boldsymbol{\pi}_A$, and tissue-type incompatibility probability θ are sufficiently small. Then the following results hold:*

- *With respect to deceased-donor/direct living-donor transplantation regime, regular-exchange regime causes steady-state transplant waiting times of all nonprioritized and unpaired patient groups and all incompatible pair groups to decrease. In particular, in addition to compatible pair groups, all incompatible overdemanded and self-demanded pair groups no longer wait for a transplant and receive exchange transplants immediately upon entry to the patient pool.*
- *With respect to regular-exchange regime, incentivized-exchange regime causes the transplant waiting times of*
  - *all overdemanded- and self-demanded-type pair groups to stay at zero,*
  - *all underdemanded-type pair groups except type $B-A$ pairs to decrease,*
  - *type $B-A$ pairs not to change,*
  - *nonprioritized and unpaired blood-type O, A, and B patient groups to decrease, and*
  - *nonprioritized and unpaired blood-type AB patient group to increase.*

Proof of Lemma 2. Suppose we fix an incentivized-exchange participation-fraction vector $\rho^* = (\rho^*_{X-Y})_{Y \triangleright X \,\&\, Y \neq X}$ such that Assumptions 2 and 3 hold. Then under any of the exchange policies (i.e., regular with $\rho = 0$ or incentivized with $\rho = \rho^*$) the flow of underdemanded-type $X - Y$ pairs and their reciprocal-type $Y - X$ pairs (from Equation 1) satisfy:

$$\boldsymbol{\pi}_{X-Y} = p_Y \alpha_X \boldsymbol{\pi}_X \geq \boldsymbol{\pi}_{Y-X} = \begin{cases} [\theta + \rho_{Y-X}(1-\theta)] p_X \alpha_Y \boldsymbol{\pi}_Y & \text{if } Y - X \neq A - B \\ p_B \alpha_A \boldsymbol{\pi}_A & \text{otherwise} \end{cases}.$$

As we have established before, in the optimal, ABO-i exchange regime for regular and incentivized exchange, none of the pairs of incompatible overdemanded and self-demanded types wait for a transplant, as they immediately receive transplant through exchange.

In the rest of the proof, we focus on the other patient groups: underdemanded-type pairs and nonprioritized and unpaired patients.

Suppose also that the tissue-type incompatibility probability, θ, and the inflow difference between types $B - A$ and $A - B$, $p_A \alpha_B \boldsymbol{\pi}_B - p_B \alpha_A \boldsymbol{\pi}_A$, are sufficiently small.

We prove the following claim first:

Claim 1. Under regular exchange, patients of all underdemanded-types pairs except that of $B - A$ are pooled with nonprioritized and unpaired deceased-donor-transplantation recipients of the same blood type, while patients of type $B - A$ pairs are never pooled with nonprioritized and unpaired blood-type B patients under any exchange regime.

Proof of Claim 1. For a blood type $X \in \{O, A, B\}$ (note that blood-type AB patients are not in any underdemanded-type pairs), under regular exchange we have $\boldsymbol{\kappa}_X\big|_{\rho=0} = 0$. We also have

$$r_X\big|_{\rho=0} = \frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X^{np\&u}\big|_{\rho=0}} > r_{X-Y}\big|_{\rho=0} = \frac{\theta p_X \alpha_Y \boldsymbol{\pi}_Y}{p_Y \alpha_X \boldsymbol{\pi}_X} \tag{9}$$

for any underdemanded type $X - Y \neq B - A$, where the inequality follows from sufficiently small θ assumption.

Recall that $k(A) = 1$ and $k(O) = 3$ are the numbers of underdemanded pair types with blood-type A and O patients, respectively.

Thus, pairs of the only underdemanded type with blood-type A patient, $A - AB$, are pooled with nonprioritized and unpaired blood-type A patients under regular exchange by Equation 9.

We order underdemanded pair types with patient blood type O according to the ascending order of their r ratios as $O - Y_1$, $O - Y_2$, and $O - Y_3$. Then, for $\ell = 1, 2$,

$$r_{O,O-Y_1,\ldots,O-Y_\ell}\big|_{\rho=0} = \frac{\boldsymbol{\delta}_O + \sum_{m=1}^{\ell} \theta p_O \alpha_{Y_m} \boldsymbol{\pi}_{Y_m}}{\boldsymbol{\pi}_O^{np\&u}\big|_{\rho=0} + \sum_{m=1}^{\ell} p_{Y_m} \alpha_O \boldsymbol{\pi}_O} > \frac{\theta p_O \alpha_{Y_{\ell+1}} \boldsymbol{\pi}_{Y_{\ell+1}}}{p_{Y_{\ell+1}} \alpha_O \boldsymbol{\pi}_O} = r_{O-Y_{\ell+1}}\big|_{\rho=0} \tag{10}$$

because of the assumption that θ is sufficiently small.

Thus, under regular exchange, underdemanded-type pairs with blood-type O patients are pooled for deceased-donor transplantation with nonprioritized and unpaired blood-type O patients.

On the other hand, for the underdemanded pair type $B - A$, we have

$$r_{B-A} = \frac{p_B \alpha_A \boldsymbol{\pi}_A}{p_A \alpha_B \boldsymbol{\pi}_B} > r_B \tag{11}$$

for any ρ because of the assumption that the difference $p_A\alpha_B\boldsymbol{\pi}_B - p_B\alpha_A\boldsymbol{\pi}_A$ is sufficiently small. Thus, pairs of type $B-A$ are never pooled with nonprioritized and unpaired blood-type B patients under regular or incentivized exchange for any ρ.

Equation 9 with $X = B$ and Equation 11 imply that pairs of type $B - AB$ are pooled with nonprioritized and unpaired blood-type B patients under regular exchange. $\square$

We also assume that the fraction of living-donor-transplant recipients reentering the patient pool, $\phi^{\mathbf{l}}$, is also sufficiently small in the rest of the proof.

Transition to Regular Exchange:

Consider a blood type X. The flow of pairs that benefit from direct living-donor transplant regime is given by $\boldsymbol{\lambda}_X = \sum_{Y:Y\triangleright X}(1-\theta)p_Y\alpha_X\boldsymbol{\pi}_X$. The flow of pairs that benefit from regular exchange satisfies $\boldsymbol{\epsilon}_X = \sum_{Y:Y\triangleright X}\theta p_Y\alpha_X\boldsymbol{\pi}_X + \sum_{Y:X\triangleright Y, Y\neq X}\theta p_X\alpha_Y\boldsymbol{\pi}_Y + \mathbf{1}_{\left\{X\in\{A,B\}\right\}}p_B\alpha_A\boldsymbol{\pi}_A > 0$.[33] This is also the flow of patients that fall out of competition from the blood-type X deceased-donor queue with respect to the deceased-donor/direct living-donor transplantation regime.

We consider the ratio of the available deceased-donor flow to the flow of patients who cannot receive direct living donation, which we refer to as $r^{\mathbf{l}}_X$, and $r_{X,X-Y_1,\ldots,X-Y_{\ell(X)|_{\rho=0}}}\big|_{\rho=0}$ when pairs of underdemanded types $X-Y_1,\ldots,X-Y_{\ell(X)|_{\rho=0}}$ are pooled for deceased-donor transplantation under regular exchange. We have

$$r^{\mathbf{l}}_X = \frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X - \underbrace{\sum_{Y:Y\triangleright X}(1-\theta)p_Y\alpha_X\boldsymbol{\pi}_X}_{=\boldsymbol{\lambda}_X} + \phi^{\mathbf{d}}\boldsymbol{\delta}_X + \phi^{\mathbf{l}}\boldsymbol{\lambda}_X} \quad \text{and} \tag{12}$$

$$r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}\big|_{\rho=0} = \frac{\boldsymbol{\delta}_X + \sum_{m=1}^{\ell(X)|_{\rho=0}}(\theta p_X\alpha_{Y_m}\boldsymbol{\pi}_{Y_m})}{\underbrace{\boldsymbol{\pi}_X - \alpha_X\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X + \phi^{\mathbf{l}}\boldsymbol{\lambda}_X + \phi^{\mathbf{l}}\boldsymbol{\epsilon}_X}_{=\boldsymbol{\pi}^{np\&u}_X\big|_{\rho=0}} + \sum_{m=1}^{\ell(X)|_{\rho=0}}(p_{Y_m}\alpha_X\boldsymbol{\pi}_X)}. \tag{13}$$

Claim 2. The transplant waiting time decreases with the addition of regular exchange to deceased-donor/direct living-donor transplantation for unpaired patients and underdemanded-type pairs.

Proof of Claim 2. For all X, we have from Equations 12 and 13 that, when $\phi^{\mathbf{l}} = 0$

$$r^{\mathbf{l}}_X\big|_{\phi^{\mathbf{l}}=0} = \frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X - (1-\theta)\sum_{Y:Y\triangleright X}p_Y\alpha_X\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X} \text{ and}$$

$$r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}\big|_{\rho=0,\phi^{\mathbf{l}}=0} = \frac{\boldsymbol{\delta}_X + \sum_{m=1}^{\ell(X)}\big|_{\rho=0,\phi^{\mathbf{l}}=0}(\theta p_X\alpha_{Y_m}\boldsymbol{\pi}_{Y_m})}{\boldsymbol{\pi}_X - (1-\sum_{m=1}^{\ell(X)}\big|_{\rho=0,\phi^{\mathbf{l}}=0}p_{Y_m})\alpha_X\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X}.$$

Since $B-A$ pairs are not pooled for deceased-donor transplantation by Claim 1, we have $B-A \neq X-Y_m$ for any X and m. Thus, for each Y_m, $X\triangleright Y_m$ and $Y_m \neq X$. Thus, we obtain

[33] The indicator function $\mathbf{1}_{\{Z\}}$gets value 1 if the event Z is true and value 0 if the event Z is false.

$1-\sum_{m=1}^{\ell(X)}\big|_{\rho=0,\phi^{\mathbf{l}}=0} p_{Y_m} \geq 1-\sum_{Y:X\triangleright Y, Y\neq X} p_Y$. We also have $1-\sum_{Y:X\triangleright Y,Y\neq X} p_Y = \sum_{Y:Y\triangleright X} p_Y > (1-\theta)\sum_{Y:Y\triangleright X} p_Y$, as $\theta>0$. Thus,

$$r^{\mathbf{l}}_X\big|_{\phi^{\mathbf{l}}=0} < r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}\big|_{\rho=0,\phi^{\mathbf{l}}=0}.$$

By the continuity of the r ratios in $\phi^{\mathbf{l}}$, for sufficiently small $\phi^{\mathbf{l}}$ we have $r^{\mathbf{l}}_X < r_X\big|_{\rho=0}$, implying that

$$t^{\mathbf{l}}_X = S^{-1}\left(r^{\mathbf{l}}_X\right) > S^{-1}\left(r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}\big|_{\rho=0}\right) = t^{\mathbf{e}}_X. \tag{14}$$

Since by Claim 1 pairs of underdemanded types except $B-A$ are pooled with deceased-donor-transplant recipients under regular exchange, their transplant waiting times also decrease. Moreover, the transplant waiting time of type $B-A$ pairs decreases even more, as it is not pooled with deceased-donor-transplant recipients by Claim 1.□

Transition to Incentivized Exchange:

Consider a blood type $X\in\{A,B,O\}$. Suppose ρ^* is the participation profile for incentivized exchange. The flow of pairs who benefit from incentivized exchange with any ρ in addition to regular exchange satisfies

$$\boldsymbol{\iota}_X = \sum_{Y:X\triangleright Y,Y\neq X} \rho_{Y-X}(1-\theta)p_X\alpha_Y\boldsymbol{\pi}_Y,$$

while the flow of prioritized reentrants satisfies

$$\phi^{\mathbf{l}}\boldsymbol{\kappa}_X = \phi^{\mathbf{l}}\left(\sum_{Y:Y\triangleright X,Y\neq X} \rho_{X-Y}(1-\theta)p_Y\alpha_X\boldsymbol{\pi}_X\right).$$

As a result, for some $\ell(X)\in\{0,\ldots,k(X)\}$, pairs of underdemanded types $X-Y_1,\ldots,X-Y_{\ell(X)}$ are pooled for deceased-donor transplantation at ρ, and thus, we have

$$r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}} = \frac{\boldsymbol{\delta}_X - \phi^{\mathbf{l}}\boldsymbol{\kappa}_X + \sum_{m=1}^{\ell(X)}\left([\theta+\rho_{Y_m-X}(1-\theta)]p_X\alpha_{Y_m}\boldsymbol{\pi}_{Y_m}\right)}{\underbrace{\boldsymbol{\pi}_X - \alpha_X\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X + \phi^{\mathbf{l}}\boldsymbol{\lambda}_X + \phi^{\mathbf{l}}\boldsymbol{\epsilon}_X + \phi^{\mathbf{l}}\boldsymbol{\iota}_X}_{=\boldsymbol{\pi}_X^{np\&u}} + \sum_{m=1}^{\ell(X)} p_{Y_m}\alpha_X\boldsymbol{\pi}_X}. \tag{15}$$

Claim 3. The transplant waiting times decrease under incentivized exchange with respect to regular exchange for nonprioritized and unpaired blood-type X patients and all underdemanded-type pairs with blood-type X patients—except type $B-A$.

Proof of Claim 3. We will show that all ratios r_{X-Y_m} for all $m=1,\ldots,k(X)$, such that $X-Y_m\neq B-A$, and ratio $r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}$ increase from $\rho=0$ to $\rho=\rho^*$, and thus, the related transplant waiting time decreases. We have $\ell(X)=\ell(X)\big|_{\rho=0}$ (i.e., the number of pooled types at $\rho=0$) for sufficiently small ρ profiles, since r ratios are continuous around $\rho=0$ and there are no sudden jumps in pooling by Claim 1. Thus, for sufficiently small ρ, when $\phi^{\mathbf{l}}=0$

$$r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}\big|_{\phi^{\mathbf{l}}=0} = \frac{\boldsymbol{\delta}_X + \sum_{m=1}^{\ell(X)|_{\phi^{\mathbf{l}}=0}}\left([\theta+\rho_{Y_m-X}(1-\theta)]p_X\alpha_{Y_m}\boldsymbol{\pi}_{Y_m}\right)}{\boldsymbol{\pi}_X - \alpha_X\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X + \sum_{m=1}^{\ell(X)|_{\phi^{\mathbf{l}}=0}} p_{Y_m}\alpha_X\boldsymbol{\pi}_X}$$

and

$$r_{X-Y_m} = \frac{[\theta + \rho_{Y_m-X}(1-\theta)]p_X\alpha_{Y_m}\boldsymbol{\pi}_{Y_m}}{p_{Y_m}\alpha_X\boldsymbol{\pi}_X}$$

are increasing in ρ. Suppose that we increase each ρ_{W-Z} from 0 to ρ^*_{W-Z} in a steady speed equal to ρ^*_{W-Z} throughout so that ρ reaches ρ^* at time $t = 1$. We can compare the rates of change in both entities along this line as the inner product of their gradient vector and speed vector:

$$\begin{aligned}\left(\nabla_\rho r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}\right)\cdot\rho^*\big|_{\phi^{\mathbf{l}}=0} &= \frac{\sum_{m=1}^{\ell(X)|_{\phi^{\mathbf{l}}=0}}\rho^*_{Y_m-X}(1-\theta)p_X\alpha_{Y_m}\boldsymbol{\pi}_{Y_m}}{\boldsymbol{\pi}_X-\alpha_X\boldsymbol{\pi}_X+\phi^{\mathbf{d}}\boldsymbol{\delta}_X+\sum_{m=1}^{\ell(X)|_{\phi^{\mathbf{l}}=0}}p_{Y_m}\alpha_X\boldsymbol{\pi}_X}\\ &< \frac{\rho^*_{Y_m-X}(1-\theta)p_X\alpha_{Y_m}\boldsymbol{\pi}_{Y_m}}{p_{Y_m}\alpha_X\boldsymbol{\pi}_X} = (\nabla_\rho r_{X-Y_m})\cdot\rho^*,\end{aligned}$$

for $m = \ell(X)|_{\phi^{\mathbf{l}}=0}$, i.e., the r ratio for the pooled nonprioritized and unpaired patients and underdemanded-type pairs changes slower than the largest of the r ratios of the underdemanded types that are pooled when $\phi^{\mathbf{l}} = 0$. Thus, as ρ increases to ρ^*, either ρ reaches ρ^* without $\ell(X)|_{\phi^{\mathbf{l}}=0}$ changing or there will be a profile ρ^1 such that $0 \ll \rho^1 < \rho^*$, at which $\ell(X)|_{\phi^{\mathbf{l}}=0}$ decreases to $\ell(X)\big|_{\rho=0,\phi^{\mathbf{l}}=0} - 1$ so that pairs of the underdemanded type with the highest r ratio are no longer pooled with the rest. Similarly, the resulting new r value relevant for the pool of nonprioritized and unpaired patients and remaining underdemanded-type pairs will be increasing in ρ until ρ reaches a new cutoff $\rho^2 \leq \rho^*$. At this new cutoff $\ell(X)\big|_{\rho=\rho^2,\phi^{\mathbf{l}}=0} = \ell(X)\big|_{\rho=0,\phi^{\mathbf{l}}=0} - 2$, and so on, so forth. Possibly, no underdemanded pairs may remain pooled at sufficiently high ρ, implying that $\ell(X)|_{\phi^{\mathbf{l}}=0} = 0$, and thus, $\left(\nabla_\rho r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}\right)\cdot\rho^*\big|_{\phi^{\mathbf{l}}=0} = 0$. Except after this last iteration, all r ratios strictly increase at each iteration until $t = 1$ at different speeds when $\phi^{\mathbf{l}} = 0$.

In the end, for sufficiently small $\phi^{\mathbf{l}}$, by the continuity of the r ratios (and their gradients) in $\phi^{\mathbf{l}}$ and by the fact that all underdemanded-type pairs were pooled initially at $\rho = 0$, all gradients are strictly positive at least for small ρ. Thus, we obtain that the r ratios strictly increase from $\rho = 0$ to $\rho = \rho^*$. As the transplant waiting time is decreasing in its relevant r ratio for each patient group, for all underdemanded types with blood-type X patient blood type—except type $B-A$—and nonprioritized and unpaired blood-type X patients, the transplant waiting times strictly decrease with respect to their levels at $\rho = 0$. □

On the other hand, all paired blood-type AB patients receive direct or exchange living-donor transplants without waiting when $\rho = 0$. This fact does not change when $\rho = \rho^*$. Thus, in both cases the flow of blood-type patients that enter the deceased-donor queue is the same. In particular, as there are no underdemanded pair types with blood-type AB patients, Equation 15 implies

$$t^{\mathbf{i}}_{AB}|_\rho = S^{-1}\left(\frac{\boldsymbol{\delta}-\phi^{\mathbf{l}}\boldsymbol{\kappa}_{AB}|_\rho}{\boldsymbol{\pi}_X-(1-\phi^{\mathbf{l}})\alpha_X\boldsymbol{\pi}_X+\phi^{\mathbf{d}}\boldsymbol{\delta}_X}\right).$$

Since $\boldsymbol{\kappa}_{AB}|_{\rho=\rho^*} > \boldsymbol{\kappa}_{AB}|_{\rho=0} = 0$,

$$t^{\mathbf{i}}_{AB}|_{\rho=\rho^*} > t^{\mathbf{i}}_{AB}|_{\rho=0},$$

i.e., the transplant waiting time of nonprioritized and unpaired blood-type AB patients strictly increases from $\rho = 0$ to $\rho = \rho^*$ regardless of $\phi^{\mathbf{l}}$ and θ.

For sufficiently small type $B-A$ and type $A-B$ flow difference, since pairs of type $B-A$ are not

pooled with nonprioritized and unpaired B patients regardless of ρ^* by Claim 1, their transplant waiting time remains unaffected for any ρ, including $\rho = 0$ and $\rho = \rho^*$. ■

A.5 Welfare Consequences of Different Transplant Regimes on Deceased-Donor Queues

Our last result of this section formulates how access to deceased-donor transplantation differs with the successive introduction of deceased-donor transplantation, living-donor transplantation, kidney exchange, and incentivized exchange.

Theorem 4 *Suppose Assumptions 2 and 3 hold. Let $p_A > p_B$. Suppose that $\alpha_X = \alpha$ for any blood type X, and $\frac{\boldsymbol{\delta}_X}{\boldsymbol{\delta}_Y} = \frac{\boldsymbol{\pi}_X}{\boldsymbol{\pi}_Y} = \frac{p_X}{p_Y}$ for any two blood types X and Y. Suppose also that the fraction of pairs taking the incentivized-exchange option is uniform at a fixed $\rho < 1$ for any eligible type. Then:*

1. *With deceased-donor transplantation only, the transplant waiting time at each deceased-donor queue is the same for any blood type X:*

$$t^{\mathbf{d}}_O = t^{\mathbf{d}}_A = t^{\mathbf{d}}_B = t^{\mathbf{d}}_{AB}.$$

2. *Introduction of direct living-donor transplantation reduces the transplant waiting time at each deceased-donor queue. The changes in transplant waiting times and the transplant waiting times are ranked as follows:*

$$(t^{\mathbf{d}}_{AB} - t^{\mathbf{l}}_{AB}) > (t^{\mathbf{d}}_A - t^{\mathbf{l}}_A) > (t^{\mathbf{d}}_B - t^{\mathbf{l}}_B) > (t^{\mathbf{d}}_O - t^{\mathbf{l}}_O),$$
$$t^{\mathbf{l}}_{max} = t^{\mathbf{l}}_O > t^{\mathbf{l}}_B > t^{\mathbf{l}}_A > t^{\mathbf{l}}_{AB} = t^{\mathbf{l}}_{min}.$$

Further suppose that θ and $\phi^{\mathbf{l}}$ are sufficiently small. Then:

3. *Introduction of kidney exchange in addition to deceased-donor/direct living-donor transplantation further reduces the transplant waiting time at each deceased-donor queue, but more for blood type B than blood type A equalizing the deceased-donor queue transplant waiting times for these two blood types. The combination of kidney exchange and living-donor transplantation reduces the transplant waiting time at the blood-type AB deceased-donor queue the most, at the blood-type A and B deceased-donor queues equally next, and at the blood-type O deceased-donor queue the least:*

$$(t^{\mathbf{d}}_{AB} - t^{\mathbf{e}}_{AB}) > (t^{\mathbf{d}}_A - t^{\mathbf{e}}_A) = (t^{\mathbf{d}}_B - t^{\mathbf{e}}_B) > (t^{\mathbf{d}}_O - t^{\mathbf{e}}_O).$$

   The inclusion of kidney exchange with deceased-donor/direct living-donor transplantation results in the following ranking of the transplant waiting times:

$$t^{\mathbf{e}}_{max} = t^{\mathbf{e}}_O > t^{\mathbf{e}}_B = t^{\mathbf{e}}_A > t^{\mathbf{e}}_{AB} = t^{\mathbf{e}}_{min}.$$

4. *Inclusion of incentivized exchange with regular exchange and deceased-donor/direct living-donor transplantation decreases the transplant waiting times at the blood-type O, A, and B deceased-donor queues but increases it at the blood-type AB deceased-donor queue. The waits at the blood-type A and B deceased-donor queues continue to be equal:*

$$t^{\mathbf{i}}_O < t^{\mathbf{e}}_O, \qquad t^{\mathbf{i}}_A = t^{\mathbf{i}}_B < t^{\mathbf{e}}_A = t^{\mathbf{e}}_B, \qquad t^{\mathbf{i}}_{AB} > t^{\mathbf{e}}_{AB}.$$

Proof of Theorem 4. Suppose Assumptions 2 and 3 hold. Let $p_A > p_B$. Suppose that $\alpha_X = \alpha$ for any blood type X, and $\frac{\boldsymbol{\delta}_X}{\boldsymbol{\delta}_Y} = \frac{\boldsymbol{\pi}_X}{\boldsymbol{\pi}_Y} = \frac{p_X}{p_Y}$ for any two blood types X and Y. Also assume that the fraction of pairs taking the incentivized-exchange option is uniform at a fixed $\rho < 1$ for any eligible pair type.

1. ***With deceased-donor transplantation only***, the transplant waiting time at each deceased-donor queue is $t^{\mathbf{d}}_X = S^{-1}\left(\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X + \phi^{\mathbf{d}}\boldsymbol{\delta}_X}\right) = S^{-1}\left(\frac{\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X}}{1+\phi^{\mathbf{d}}\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X}}\right)$ for any blood type X. Since $\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X} = \frac{\boldsymbol{\delta}_Y}{\boldsymbol{\pi}_Y}$ for any two blood types X and Y, we have $t^{\mathbf{d}}_X = t^{\mathbf{d}}_Y$.
2. ***Introduction of direct living-donor transplantation*** reduces the transplant waiting time at each deceased-donor queue X, since $t^{\mathbf{l}}_X = S^{-1}\left(\frac{\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X}}{1+\phi^{\mathbf{d}}\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X} - (1-\phi^{\mathbf{l}})p^{\mathbf{l}}_X\alpha}\right) < S^{-1}\left(\frac{\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X}}{1+\phi^{\mathbf{d}}\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X}}\right) = t^{\mathbf{d}}_X$. Since the probability of being compatible with the paired donor conditional on having a living donor satisfies for each blood type

$$p^{\mathbf{l}}_O = (1-\theta)p_O, \qquad p^{\mathbf{l}}_B = (1-\theta)(p_O + p_B),$$
$$p^{\mathbf{l}}_A = (1-\theta)(p_O + p_A), \qquad p^{\mathbf{l}}_{AB} = (1-\theta)(p_O + p_A + p_B + p_{AB}) = (1-\theta),$$

and $p_A > p_B$, we have $p^{\mathbf{l}}_O < p^{\mathbf{l}}_B < p^{\mathbf{l}}_A < p^{\mathbf{l}}_{AB}$. Thus, as $t^{\mathbf{l}}_X$ is decreasing in $p^{\mathbf{l}}_X$ and $\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X}$ is constant among blood types, we have

$$t^{\mathbf{l}}_{AB} < t^{\mathbf{l}}_A < t^{\mathbf{l}}_B < t^{\mathbf{l}}_O.$$

Moreover, Part 1 implies that

$$\left(t^{\mathbf{d}}_{AB} - t^{\mathbf{l}}_{AB}\right) > \left(t^{\mathbf{d}}_A - t^{\mathbf{l}}_A\right) > \left(t^{\mathbf{d}}_B - t^{\mathbf{l}}_B\right) > \left(t^{\mathbf{d}}_O - t^{\mathbf{l}}_O\right).$$

Further assume that θ and $\phi^{\mathbf{l}}$ are sufficiently small in the rest of the proof. We also have the flow difference between type $B-A$ and type $A-B$ as $p_B\alpha\boldsymbol{\pi}_A - p_A\alpha\boldsymbol{\pi}_B = 0$ since $\frac{p_A}{p_B} = \frac{\boldsymbol{\pi}_A}{\boldsymbol{\pi}_B}$. Thus, hypothesis of Lemma 2 holds.

3. ***Introduction of regular exchange***, in addition to deceased-donor/direct living-donor transplantation, causes the deceased-donor waiting times for all blood types to decrease by Lemma 2. By Claim 1 in the proof of the same lemma, pairs of all underdemanded types except $B-A$ are pooled for deceased-donor transplantation with unpaired patients of their patients' blood types. By Equation 13 and $\frac{\boldsymbol{\pi}_X}{\boldsymbol{\pi}_Y} = \frac{p_X}{p_Y}$ for any two blood types X, Y, we obtain

$$\begin{aligned} r_{O,O-A,O-B,O-AB}\big|_{\rho=0} &= \frac{\boldsymbol{\delta}_O + (\theta p_O\alpha\boldsymbol{\pi}_A + \theta p_O\alpha\boldsymbol{\pi}_B + \theta p_O\alpha\boldsymbol{\pi}_{AB})}{\boldsymbol{\pi}_O - \alpha\boldsymbol{\pi}_O + \phi^{\mathbf{d}}\boldsymbol{\delta}_O + \phi^{\mathbf{l}}(\boldsymbol{\lambda}_O + \boldsymbol{\epsilon}_O) + (p_A\alpha\boldsymbol{\pi}_O + p_B\alpha\boldsymbol{\pi}_O + p_{AB}\alpha\boldsymbol{\pi}_O)} \\ &= \frac{\frac{\boldsymbol{\delta}_O}{\boldsymbol{\pi}_O} + (\theta p_A\alpha + \theta p_B\alpha + \theta p_{AB}\alpha)}{1 - \alpha + \phi^{\mathbf{d}}\frac{\boldsymbol{\delta}_O}{\boldsymbol{\pi}_O} + \phi^{\mathbf{l}}\left(\frac{\boldsymbol{\lambda}_O + \boldsymbol{\epsilon}_O}{\boldsymbol{\pi}_O}\right) + (p_A\alpha + p_B\alpha + p_{AB}\alpha)}, \end{aligned}$$

$$r_{A,A-AB}\big|_{\rho=0} = \frac{\boldsymbol{\delta}_A + (\theta p_A \alpha \boldsymbol{\pi}_{AB})}{\boldsymbol{\pi}_A - \alpha\boldsymbol{\pi}_A + \phi^{\mathbf{d}}\boldsymbol{\delta}_A + \phi^{\mathbf{l}}(\boldsymbol{\lambda}_A + \boldsymbol{\epsilon}_A) + (p_{AB}\alpha\boldsymbol{\pi}_A)} = \frac{\frac{\boldsymbol{\delta}_A}{\boldsymbol{\pi}_A} + (\theta p_{AB}\alpha)}{1 - \alpha + \phi^{\mathbf{d}}\frac{\boldsymbol{\delta}_A}{\boldsymbol{\pi}_A} + \phi^{\mathbf{l}}\left(\frac{\boldsymbol{\lambda}_A + \boldsymbol{\epsilon}_A}{\boldsymbol{\pi}_A}\right) + (p_{AB}\alpha)},$$

$$r_{B,B-AB}\big|_{\rho=0} = \frac{\boldsymbol{\delta}_B + (\theta p_B \alpha \boldsymbol{\pi}_{AB})}{\boldsymbol{\pi}_B - \alpha\boldsymbol{\pi}_B + \phi^{\mathbf{d}}\boldsymbol{\delta}_B + \phi^{\mathbf{l}}(\boldsymbol{\lambda}_B + \boldsymbol{\epsilon}_B) + (p_{AB}\alpha\boldsymbol{\pi}_B)} = \frac{\frac{\boldsymbol{\delta}_B}{\boldsymbol{\pi}_B} + (\theta p_{AB}\alpha)}{1 - \alpha + \phi^{\mathbf{d}}\frac{\boldsymbol{\delta}_B}{\boldsymbol{\pi}_B} + \phi^{\mathbf{l}}\left(\frac{\boldsymbol{\lambda}_B + \boldsymbol{\epsilon}_B}{\boldsymbol{\pi}_B}\right) + (p_{AB}\alpha)},$$

$$r_{AB} = \frac{\boldsymbol{\delta}_{AB}}{\boldsymbol{\pi}_B - \alpha\boldsymbol{\pi}_{AB} + \phi^{\mathbf{d}}\boldsymbol{\delta}_{AB} + \phi^{\mathbf{l}}(\boldsymbol{\lambda}_{AB} + \boldsymbol{\epsilon}_{AB})} = \frac{\frac{\boldsymbol{\delta}_{AB}}{\boldsymbol{\pi}_{AB}}}{1 - \alpha + \phi^{\mathbf{d}}\frac{\boldsymbol{\delta}_{AB}}{\boldsymbol{\pi}_{AB}} + \phi^{\mathbf{l}}\left(\frac{\boldsymbol{\lambda}_{AB} + \boldsymbol{\epsilon}_{AB}}{\boldsymbol{\pi}_{AB}}\right)}.$$

Since $\frac{\boldsymbol{\pi}_X}{\boldsymbol{\pi}_Y} = \frac{\boldsymbol{\delta}_X}{\boldsymbol{\delta}_Y}$ for any two blood types X and Y, we have by Theorem 3, $\frac{\boldsymbol{\lambda}_A + \boldsymbol{\epsilon}_A}{\boldsymbol{\pi}_A} = \frac{\boldsymbol{\lambda}_B + \boldsymbol{\epsilon}_B}{\boldsymbol{\pi}_B}$, and thus, $r_{A,A-AB}\big|_{\rho=0} = r_{B,B-AB}\big|_{\rho=0}$ implying that

$$t^{\mathbf{e}}_A = S^{-1}\left(r_{A,A-AB}\big|_{\rho=0}\right) = S^{-1}\left(r_{B,B-AB}\big|_{\rho=0}\right) = t^{\mathbf{e}}_B.$$

Suppose $\phi^{\mathbf{l}} = 0$. Then,

$$r_{O,O-A,O-B,O-AB}\big|_{\rho=0,\phi^{\mathbf{l}}=0} = \frac{\frac{\boldsymbol{\delta}_O}{\boldsymbol{\pi}_O} + (\theta p_A\alpha + \theta p_B\alpha + \theta p_{AB}\alpha)}{1 - \alpha + \phi^{\mathbf{d}}\frac{\boldsymbol{\delta}_O}{\boldsymbol{\pi}_O} + (p_A\alpha + p_B\alpha + p_{AB}\alpha)},$$

$$r_{A,A-AB}\big|_{\rho=0,\phi^{\mathbf{l}}=0} = \frac{\frac{\boldsymbol{\delta}_A}{\boldsymbol{\pi}_A} + (\theta p_{AB}\alpha)}{1 - \alpha + \phi^{\mathbf{d}}\frac{\boldsymbol{\delta}_A}{\boldsymbol{\pi}_A} + (p_{AB}\alpha)}, \text{ and}$$

$$r_{AB}\big|_{\phi^{\mathbf{l}}=0} = \frac{\frac{\boldsymbol{\delta}_{AB}}{\boldsymbol{\pi}_{AB}}}{1 - \alpha + \phi^{\mathbf{d}}\frac{\boldsymbol{\delta}_{AB}}{\boldsymbol{\pi}_{AB}}}.$$

Since for sufficiently small θ, $\frac{\boldsymbol{\delta}_X/\boldsymbol{\pi}_X}{1-\alpha+\phi^{\mathbf{d}}\boldsymbol{\delta}_X/\boldsymbol{\pi}_X} > \theta$ for any X, we have that $\frac{\boldsymbol{\delta}_X/\boldsymbol{\pi}_X+\theta q(X)}{1-\alpha+\phi^{\mathbf{d}}\boldsymbol{\delta}_X/\boldsymbol{\pi}_X+q(X)}$ is decreasing in $q(X)$ for any $q(X) \geq 0$. Thus, we can rank the above entities as $r_{O,O-A,O-B,O-AB}\big|_{\rho=0,\phi^{\mathbf{l}}=0} < r_{A,A-AB}\big|_{\rho=0,\phi^{\mathbf{l}}=0} < r_{AB}\big|_{\phi^{\mathbf{l}}=0}$. By the continuity of these r ratios in $\phi^{\mathbf{l}}$, for sufficiently small $\phi^{\mathbf{l}}$ we still have $r_{O,O-A,O-B,O-AB}\big|_{\rho=0} < r_{A,A-AB}\big|_{\rho=0} < r_{AB}$. As the generic transplant waiting time $t = S^{-1}(r)$ is decreasing in r, we can rank the waiting times for deceased-donor transplantation in the queue under regular exchange as

$$t^{\mathbf{e}}_{AB} < t^{\mathbf{e}}_A = t^{\mathbf{e}}_B < t^{\mathbf{e}}_O,$$

and thus, by Part 1,

$$\left(t^{\mathbf{d}}_{AB} - t^{\mathbf{e}}_{AB}\right) > \left(t^{\mathbf{d}}_A - t^{\mathbf{e}}_A\right) = \left(t^{\mathbf{d}}_B - t^{\mathbf{e}}_B\right) > \left(t^{\mathbf{d}}_O - t^{\mathbf{e}}_O\right).$$

4. ***Introduction of incentivized exchange***, in addition to deceased-donor/direct living-donor transplantation and regular exchange, causes the waiting time for a deceased-donor transplant to decrease for all blood types except AB, for which it increases by Lemma 2. Since $\frac{p_X}{p_Y} = \frac{\boldsymbol{\pi}_X}{\boldsymbol{\pi}_Y}$ for any two blood types X and Y, the relevant r ratios for transplant waiting times in the deceased-donor queue satisfy for each $k = 0, \ldots, k(X)$, such that $X - Y_k \neq B - A$,

$$r_{X,X-Y_1,\ldots,X-Y_k} = \frac{\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X} - \phi^{\mathbf{l}}\frac{\boldsymbol{\kappa}_X}{\boldsymbol{\pi}_X} + \sum_{m=1}^{k}\left([\theta + \rho(1-\theta)]p_{Y_m}\alpha\right)}{1 - \alpha + \phi^{\mathbf{d}}\frac{\boldsymbol{\delta}_X}{\boldsymbol{\pi}_X} + \phi^{\mathbf{l}}\frac{\boldsymbol{\lambda}_X + \boldsymbol{\epsilon}_X + \boldsymbol{\iota}_X}{\boldsymbol{\pi}_X} + \sum_{m=1}^{k} p_{Y_m}\alpha},$$

where $\frac{\boldsymbol{\lambda}_O + \boldsymbol{\epsilon}_O + \boldsymbol{\iota}_O}{\boldsymbol{\pi}_O} < \frac{\boldsymbol{\lambda}_A + \boldsymbol{\epsilon}_A + \boldsymbol{\iota}_A}{\boldsymbol{\pi}_A} = \frac{\boldsymbol{\lambda}_B + \boldsymbol{\epsilon}_B + \boldsymbol{\iota}_B}{\boldsymbol{\pi}_{AB}} < \frac{\boldsymbol{\lambda}_{AB} + \boldsymbol{\epsilon}_{AB} + \boldsymbol{\iota}_B}{\boldsymbol{\pi}_B} = \alpha$ by Theorem 3, and

$$\frac{\boldsymbol{\kappa}_O}{\boldsymbol{\pi}_O} = 0 \quad < \frac{\boldsymbol{\kappa}_A}{\boldsymbol{\pi}_A} = \rho(1-\theta)p_O\alpha \quad = \frac{\boldsymbol{\kappa}_B}{\boldsymbol{\pi}_B} = \rho(1-\theta)p_O\alpha \quad < \frac{\boldsymbol{\kappa}_{AB}}{\boldsymbol{\pi}_{AB}} = \rho(1-\theta)(p_O + p_A + p_B)\alpha.$$

Moreover, we have that for all underdemanded types $X-Y$ except type $B-A$, the r ratio

$$r_{X-Y} = \frac{[\theta+\rho(1-\theta)]p_X\alpha\boldsymbol{\pi}_Y}{p_Y\alpha\boldsymbol{\pi}_X} = \theta+\rho(1-\theta) \tag{16}$$

is uniform. Define $\hat{r}_X := r_{X,X-Y_1,\ldots,X-Y_{\ell(X)}}$. Thus, type $A-AB$ pairs will be pooled with nonprioritized and unpaired blood-type A patients if and only if type $B-AB$ pairs will be pooled with nonprioritized and unpaired blood-type B patients. This implies $\hat{r}_A = \hat{r}_B$ and

$$t^{\mathbf{i}}_A = S^{-1}(\hat{r}_A) = S^{-1}(\hat{r}_B) = t^{\mathbf{i}}_B.$$

■

Appendix B Construction of Calibration Parameters for Numerical Predictions of the Model

In this appendix, we explain how the calibration parameters, reported in Table 2 in Section 5 and used to generate the numerical model predictions, are constructed.

In Table 5, we report the blood-type distribution for different ethnicities and fractions of these ethnicities in the US population. Using these, we calculate an overall US blood-type distribution (the last row of this table). We use this as the blood-type distribution of living donors, (p_X), in our model.

US Blood Type and Ethnicity Distribution Data					
Ethnicities	**Blood Types**				**Pop. %**
	O	*A*	*B*	*AB*	
African American	0.490	0.270	0.200	0.040	12.4%
Asian American	0.400	0.280	0.270	0.050	3.3%
Native American	0.790	0.160	0.040	0.010	0.8%
White American	0.450	0.400	0.110	0.040	83.4%
US population	0.456	0.378	0.126	0.040	

Table 5: The US blood type and ethnicity distribution retrieved from `http://bloodbook.com` (on 03/05/2018). The blood-type distribution for the overall US population is constructed using the ethnicity distribution and could be slightly different from the general distributions reported in other sources.

In Table 6, we report the OPTN/SRTR data for average of deceased-donor queue additions and deceased- and living-donor transplants between 2009-2017. We measure time in one year units and calculate the flows using the annual numbers reported in this data. First, we observe that on average $\frac{2\times 7936}{11714} = 1.4761$ kidneys are harvested from each deceased donor, since a total of 7936 deceased donors arrive while 11714 deceased-donor transplants are conducted. The deceased-donor flows, $(\boldsymbol{\delta}_X)$, are constructed by multiplying each entry in the second to last row of the table with 1.4761. The row above, deceased-donation recipients, is used as the de-facto deceased-donor flows, $(\boldsymbol{\delta}'_X)$, in the numerical calculations.

The US OPTN/SRTR Kidney Data						
		O	*A*	*B*	*AB*	All
Patient Arrivals	Total Additions to the Queue	17,010	11,507	5,156	1,337	35,009
	Living-Donor-Transplant Recipients not on Queue	179	173	52	17	420
	Reentrants	1,973	1,481	592	181	4,227
Total Transplants	Direct Living-Donor Transplants	2,127	1,978	667	208	4,979
	Other Living-Donor Transplants	421	283	112	27	842
	Deceased-Donor Transplants	5,357	4,188	1,548	621	11,714
Deceased-Donor Arrivals		3,786	2,942	939	268	7,936
Average CPRA (for the year 2017)						6.79%

Table 6: Arrival and transplant averages per year to and from the kidney deceased-donor queue for 2009-2017 entrants. Data is obtained from OPTN/SRTR using the "advanced report" option from `http://optn.transplant.hrsa.gov` (on 10/30/2018).

New patient arrival flows, $(\boldsymbol{\pi}_X)$, are calculated as follows: We know the annual additions to the deceased-donor queue (the first row of the table). However, some patients receive living-donor transplants without even registering in the queue (the second row of the table). We add these two numbers and subtract the number of reentrants (the third row of the table) from them to find $\boldsymbol{\pi}_X$ for each blood type X.

Reentry fractions, $\phi^{\mathbf{l}}$ and $\phi^{\mathbf{d}}$, are assumed to be the same, as the OPTN/SRTR national data do not distinguish reentrants based on their previous transplantation type. We divide the total number of reentrants (the last cell of the third row of the table) by the total number of transplants (the sum of the last cells of the fourth-sixth rows of the table).

The tissue-type incompatibility probability, θ, is taken as the average calculated panel reactive antibody (CPRA), 0.0679, for the 2017 entrants (see Table 10 in Appendix D for its calculation using the OPTN/SRTR data). CPRA measures the percentage of the population with which the patient is tissue-type incompatible. We chose year 2017 because the entry flow CPRA has been increasing over time since 2009. We consider different θ values in our robustness analyses as explained in in Section 5.2.

The calculation of paired-donor fractions, (α_X), requires the knowledge of the total number of patients who arrive with paired donors. However, this information is not available since only the realized living-donor transplants are recorded in this database. Most of these transplants are direct transplants, i.e., those from the compatible paired donor of a patient. A smaller percentage of those are from exchanges or from non-directed altruistic living donors. In the fourth row of Table 6, we report the numbers of direct living-donor transplants conducted (i.e., each entry is $\boldsymbol{\lambda}_X$ in our model). Assuming patients and living donors are paired initially as in our model, we calculate the probability of having a compatible donor conditional on being paired with a living donor. These probabilities are calculated as follows using the living-donor blood-type distribution, (p_X), reported

in the last row of Table 5:

$$p_O^{\mathbf{l}} = (1-\theta)p_O = 0.4251, \qquad p_B^{\mathbf{l}} = (1-\theta)(p_O + p_B) = 0.5424,$$
$$p_A^{\mathbf{l}} = (1-\theta)(p_O + p_A) = 0.7773, \text{ and} \qquad p_{AB}^{\mathbf{l}} = (1-\theta) = 0.9321.$$

Then, we calculate $\alpha_X = \frac{\boldsymbol{\lambda}_X}{p_X^{\mathbf{l}} \boldsymbol{\pi}_X}$ for each blood type X. These values are stated in Table 2.

The incentivized-exchange participation fraction for a compatible pair type $X - Y$ with $Y \triangleright X$ and $Y \neq X$, ρ_{X-Y}, is our free calibration variable. We assume that this fraction is uniform for each type, and we denote it as ρ. We consider five regimes with $\rho = 10$, 20, 30, 50, and 100 percent.

The calibration of the survival function is explained in Appendix C.

Appendix C Calibrating Transplant Waiting Times

In this appendix, we give the model calibration results under benchmark parameters regarding transplant waiting times, using the analytical derivations in Appendix A. We start with the survival function and then give the results using this survival function.

C.1 Calibration of the Survival Function

Survival rate function $S(t)$ is obtained from OPTN/SRTR for deceased-donor queue departures. We fit a piecewise linear function (for t measured in years) as

$$S(t) = \sum_{k=1}^{K} \mathbf{1}_{\{t_{k-1} \leq t < t_k\}} \cdot \left(\frac{t_k - t}{t_k - t_{k-1}} S_{k-1} + \frac{t - t_{k-1}}{t_k - t_{k-1}} S_k \right)$$

with indicator function $\mathbf{1}_{\{Z\}}$ getting value one if Z is a true event and value zero otherwise. We used the anchor survival rates $S_1, \ldots, S_6$ and times $t_1, \ldots, t_6$ reported in Table 7.

Survival Rates in the Deceased-Donor Queue						
Time in years (t_k):	0.5	1	1.5	2	2.5	3
Surviving Fraction (S_k):	97.1%	94.5%	89.8%	83.0%	78.1%	70.0%

Table 7: For 2013 entrants obtained from Figure K16 in OPTN/SRTR "2016 Annual Data Report: Kidney" (Hart et al., 2018) through the following calculation: the ratio of the patients on the deceased-donor queue to the total number of patients who did not receive transplant at the end of each of the time periods reported above.

We use $t_0 = 0$ and $S_0 = 100\%$ for the initial anchors. We only have data for three years. For the final interval at between $t_6 = 3$ and t_7, we assume the same slope as in the previous interval from $t_5 = 2.5$ to $t_6 = 3$ continues (which is a slope of -15% per year). Thus, we obtain $S_7 = 0\%$ at $t_7 = 7.32$ years.

C.2 Numerical Predictions of the Model: Transplant Waiting Times

Table 8 reports the numerical predictions of our model for waiting times for nonprioritized deceased-donor transplantation across all regimes. These waiting times are calculated conditional on receiving a transplant.

A more standard waiting time measure used by SRTR in the US is the **median transplant waiting time**, which is the time at which half of the patients of a given cohort have received a transplant.[34] Some of the reported waiting times in Table 8 also correspond to the median transplant waiting time; those are the ones reported in boldface. Other regimes do not have well-defined median transplant waiting times because more than half of the patients of a steady-state cohort die without a transplant. As seen in the last column of Table 4 in Section 5.1, always less than 50 percent of B blood-type patients receive transplants under any regime. Therefore, there is no median transplant waiting time defined for them. For the general patient population, median transplant time is well defined starting with the availability of regular exchange. At this regime, the overall service rate is 50.1 percent of all new and reentering patients. As noted before, this finding is consistent with the current situation in the US, in which since 2005, no yearly cohort has an assigned overall median transplant waiting time empirically. The median patient of 2005 cohort is still in the deceased-donor queue as of December 2018. Regular exchange technologies are not currently fully utilized in their full extent in the US (see Agarwal et al., 2018). Our model predicts in such cases median transplant waiting time is not well defined.

Numerical Predictions of the Model:
Time to Nonprioritized Deceased-Donor Transplant
Conditional on Receiving One / Unconditional
Median Time to Any Kind of Transplant (only in boldface)

	O	*A*	*B*	*AB*	All	*O*	*A*	*B*	*AB*	All
Deceased-Donor Transplantation Only						**All plus $\rho = 20\%$-Incentivized E.**				
ABO-i	5.24	4.94	5.60	5.40	5.18	4.83	**4.51**	5.24	5.16	**4.77**
De-facto	5.32	5.02	5.41	4.43	5.18	4.91	**4.60**	5.01	**4.06**	**4.77**
Deceased/Direct Living						**All plus $\rho = 30\%$-Incentivized E.**				
ABO-i	5.02	**4.58**	5.40	5.13	4.90	4.76	**4.52**	5.25	5.19	**4.75**
De-facto	5.11	**4.66**	5.20	**4.04**	4.90	4.85	**4.61**	5.03	**4.09**	**4.75**
Deceased/Direct Living & Exchange						**All plus $\rho = 50\%$-Incentivized E.**				
ABO-i	4.95	**4.47**	5.20	5.11	**4.81**	**4.65**	**4.56**	5.29	5.24	**4.71**
De-facto	5.04	**4.56**	4.97	**4.01**	**4.81**	4.74	**4.62**	5.07	**4.14**	**4.71**
All plus $\rho = 10\%$-Incentivized E.						**All plus $\rho = 100\%$-Incentivized E.**				
ABO-i	4.89	**4.49**	5.22	5.14	**4.79**	**4.58**	**4.62**	5.37	5.38	**4.71**
De-facto	4.98	**4.58**	4.99	**4.03**	**4.79**	**4.68**	**4.71**	5.14	**4.27**	**4.73**

Table 8: Numerical predictions of the model for waiting times for nonprioritized deceased-donor transplantation *conditional* on receiving this type of a transplant (measured in years) under the benchmark parameters. The transplant waiting times in boldface also refer the (*unconditional*) **median waiting time** to either deceased-donor or living-donor transplant, i.e., the time at which half of a cohort have received transplants. If an entry is not bold, it means that the median patient dies, i.e., less than half of a cohort receive transplants, and thus, a median transplant waiting time cannot be calculated.

Nonprioritized deceased-donor-transplant waiting times (and thus, overall median transplant waiting times) decrease with increasing participation rates of pairs to incentivized exchange. For

[34] See for example `https://www.srtr.org/about-the-data/guide-to-key-transplant-program-metrics/txguidearticles/time-to-transplant/` (retrieved on 12/13/2018).

example, for each additional 10 percent participation increase, the overall transplant waiting time decreases about 0.04 years or two weeks up to $\rho = 50$ percent.

The largest waiting-time gap in only-deceased-donor transplantation regime with de-facto allocation is between types B and AB, as 0.98 years (see Table 8). This gap further increases to 1.16 years in the deceased-donor/direct living-donor transplantation regime. Addition of regular exchange decreases the largest gap to 1.03 years (though for this regime the largest gap is between types O and AB). For each $\Delta\rho = 10$ percent increase in participation in incentivized exchange further decreases the largest gap by about 6 days (i.e., 0.016 years) (which is between types B and AB when incentivized exchange becomes available). Thus, besides its welfare improving effects, incentivized exchange seems to alleviate also the transplant waiting time inequality across blood types.

Waiting times for living-donor-transplant recipients conditional on receiving a transplant are reported in Table 9. For overdemanded and self-demanded pair types, waiting time for a living-donor transplant is always zero. For underdemanded pair types, with increasing participation to incentivized exchange, the transplant waiting times weakly decrease. For low participation rates, most types are pooled with nonprioritized and unpaired patients and receive transplants at the same time with nonprioritized deceased-donor-transplant recipients. On the other hand, they are no longer pooled with nonprioritized and unpaired patients under full participation. An exception is $B - A$. Pairs of this type get matched exclusively with $A - B$ pairs as long as some form of exchange is feasible. They wait for only 3.06 years with the availability of exchange.

Numerical Predictions of the Model*: *Time to Transplant for Blood-Type-Incompatible Pairs Conditional on Receiving a Transplant

		$O-A$	$O-B$	$O-AB$	$A-B$	$A-AB$	$B-A$	$B-AB$
Deceased/Direct Living		pooled	pooled	pooled	pooled	pooled	pooled	pooled
Deceased/Direct Living & Exchange		pooled	pooled	pooled	0	pooled	3.06	pooled
All	$\rho = 10\%$	pooled	pooled	pooled	0	pooled	3.06	pooled
plus	$\rho = 20\%$	pooled	pooled	pooled	0	pooled	3.06	pooled
Incentivized	$\rho = 30\%$	pooled	pooled	pooled	0	pooled	3.06	pooled
Exchange	$\rho = 50\%$	pooled	4.39	pooled	0	pooled/4.60	3.06	pooled
	$\rho = 100\%$	3.53	1.56	4.20	0	2.05	3.06	3.81

Table 9: Numerical predictions of the model for time to transplant for blood-type-incompatible pair types (measured in years) *conditional* on receiving a transplant. "Pooled" means type $X - Y$ pairs are pooled with nonprioritized and unpaired blood-type X patients under both ABO-i and de-facto deceased-donor allocation policies. Thus, these paired patients wait as long as their nonprioritizded and unpaired counterparts and receive either a deceased-donor transplant or a living-donor transplant through exchange (only if exchange is available) at this time. One exception is noted: Type $A - AB$ pairs are pooled with nonprioritizded and unpaired blood-type A patients under ABO-i policy. On the other hand, after waiting for 4.60 years they receive entirely living-donor transplants, as long as they can live that long, under de-facto policy. Note that the "pooled" transplant waiting times are the same times as the ones reported in Table 8 for the respective patient blood types.

Appendix D Simulations

In addition to the numerical model predictions in Section 5, we also conduct simulations emulating the discrete paired- and unpaired-patient and deceased-donor arrival processes in real life. Our goal in conducting these simulations is to assess the welfare and equity consequences of our policy proposal, incentivized exchange, more accurately. Moreover, the simulations give us a chance to assess the validity of our continuum model in conducting policy analysis. We also assess the impact of alternative exchange technologies, such as three-way exchange in addition to two-way.

D.1 Simulation Methodology

In the simulations, we allocate deceased-donor kidneys according to the de-facto allocation policy on a FIFO basis to a compatible patient. If no compatible patient is found in the queue, the kidney immediately perishes. We evaluate our proposal under two exchange-size restrictions, two-way exchange and two-and-three-way exchange: Each arriving eligible type $X-Y$ pair waits to match in the next run of the kidney-exchange mechanism. The exchange is run once in every month, and, hence, 12 times a year. As most real-life kidney-exchange programs do, the exchange mechanism myopically maximizes the number of transplants among the available pairs using the considered exchange-size policy. It chooses one arbitrary maximum matching and implements it. If a compatible pair that is participating in incentivized exchange cannot be matched after one exchange run, then it is taken out of the exchange pool. In this case, the patient of the pair receives a direct transplant. Nevertheless, the patient of such a pair is eligible for a prioritized deceased-donor transplant if he reenters the patient pool.

We assume that patients are heterogenous in their tissue-type incompatibility probabilities. We use the entrant CPRA distribution reported in Table 10 to generate the tissue-type incompatibility probability θ_i for each patient i. The mean of this distribution gives us the value of θ we use in the benchmark numerical model predictions, 0.068 (or 6.8 percentage points in the CPRA reporting metric), in Section 5. This table gives the fraction of entrants in five different CPRA intervals. We assume that all patients uniformly and randomly take CPRA values in their assigned CPRA intervals. For example, this table reports that 4.25 percent of all entrants have CPRA points between 0 percent and 20 percent (the second column of this table). We first assume that a simulated patient i is assigned to this group with probability 0.0425. Then his exact tissue-type incompatibility probability θ_i is determined uniformly randomly from the interval $(0, 0.2)$.

The US OPTN/SRTR Data for CPRA Distribution for Entrants					
	CPRA intervals (in % points)				
	0	**(0,20)**	**[20,80)**	**[80,98)**	**[98,100)**
Fraction of Entrants	86.35%	4.25%	5.53%	2.22%	1.64%

Table 10: Data is obtained from the OPTN/SRTR using the "national data" option for the year 2017 from http://optn.transplant.hrsa.gov(on 10/30/2018).

Our simulations use a scaled-down version of the calibrated inflow rates for new patients and

deceased donors. The US consists of 13 transplant regions of various sizes. Deceased-donor kidneys are first offered to patients within their arrival regions. If a suitable match cannot be found in the region, then they are offered nationally. Our simulation roughly maps to one small region that comprises 1/20 of the population of the US and reflects the same patient and donor characteristics as the overall US population does. Thus, we obtain deceased-donor and new-patient arrival flows by dividing the population flows $\boldsymbol{\delta}'_X$ and $\boldsymbol{\pi}_X$ reported in Table 2 by 20. For the other parameters of the simulation, (p_X), (α_X), $\phi^{\mathbf{l}}$, $\phi^{\mathbf{d}}$, and $S(t)$, we use the same parameters reported in this table and Table 7 in Appendix C, respectively.

In each iteration, we simulate the evolution of the kidney-allocation process in such a region for 15 years.[35] Each year is divided into finite periods so that in each period either only one new patient, reentrant, or deceased donor arrives. Thus, the number of periods in each year equals the sum of the total flow of new patients, $\sum_X \boldsymbol{\pi}_X/20$, the total flow of deceased-donor kidneys, $\sum_X \boldsymbol{\delta}'_X/20$, and the total number of reentrants. The number of reentrants in a year is calculated as the minimum of (a) the reentry fraction ϕ multiplied by the number of total transplants in the previous year and (b) the total number of patients who previously received a transplant and are still alive. The numbers of patients waiting in the queue, periods per year, and reentrants per year stabilize after a number of years passes. We report the averages of the last three years (years $13-15$). We run a total of 100 simulations and report their averages and standard errors.

The new-patient, deceased-donor, and reentrant generation processes are as follows: Each new patient is generated independently and randomly with the underlying blood-type, tissue-type incompatibility probability, and living-donor pairing probability distributions. We also randomly determine his survival time while waiting for a transplant so that the population probability of remaining alive after waiting for t years is $S(t)$. Once a patient is deemed paired, his paired donor's blood type is also independently and randomly generated in a similar fashion using the living-donor blood-type distribution. We determine whether they are compatible using their blood types and the patient's tissue-type incompatibility probability with a random donor. For a deceased-donor kidney, we only generate its blood type according to the distribution dictated by $(\boldsymbol{\delta}'_X/20)$. A reentrant to the patient pool is determined according to the reentry probability among the living transplanted patients with uniform distribution. We use the following transplanted patient survival functions to determine how long each patient lives after receiving a transplant:

- The living-donor-transplant recipient survival-probability function is $S^{\mathbf{l}}(t) = 1.00e^{-0.033t}$.[36]
- The deceased-donor-transplant recipient survival-probability function is $S^{\mathbf{d}}(t) = 0.99e^{-0.050t}$.[37]

They are obtained by non-linear least squares using the survival probabilities reported in Table 11 including the survival rate 100 percent for $t = 0$.

[35] Note that according to our survival function a patient can remain alive at most for 7.7 years without receiving a transplant, as reported in Appendix C.

[36] Using NLLS, the coefficients' 95% confidence intervals are $(0.9893, 1.011)$ for 1.00 and $(-0.0378, -0.0286)$ for -0.033. We also have $R^2 = 0.9905$.

[37] Using NLLS, the coefficients' 95% confidence intervals are $(0.9773, 1.008)$ for 0.99 and $(-0.05654, -0.04286)$ for -0.050. We also have $R^2 = 0.9909$.

US Transplant Survival Rates					
Time:	3 mo.	1 yr.	2 yr.	3 yr.	5 yr.
Living-Donor Transplant Recipient	98.9%	96.3%	94.3%	91.2%	84.1%
Deceased-Donor Transplant Recipient	97.1%	93.9%	90.4%	86.4%	76.8%

Table 11: The reported survival rate are for patients who received transplants in 2011 and obtained from the "2018 USRDS Annual Data Report" (United States Renal Data System, 2018), Volume 2, Table 5.3 of Chapter 5 (retrieved from `https://www.usrds.org/2018/view/Default.aspx` on 10/30/2018).

For a reentrant, we use his original tissue-type incompatibility probability and blood type. We assume that he is now unpaired. We also randomize his new survival time using the same overall survival probability function $S(t)$.

We consider 13 regimes in our simulations. The first eight regimes are (1) only deceased-donor transplantation, (2) deceased-donor/direct living-donor transplantation, (3) deceased-donor/direct living-donor transplantation and regular exchange, and (4, 5, 6, 7, 8) deceased-donor/direct living-donor transplantation, regular and incentivized exchange for uniform participation rates $\rho = 10$, 20, 30, 50, 100 percent. These are also used in our numerical model predictions. We also consider five additional incentivized-exchange regimes in which compatible type $X-X$ pairs are also incentivized. In our continuum model, this incentivization scheme does not have additional welfare benefits, as all incompatible type $X - X$ pairs are matched with each other in regular exchange as soon as they arrive. On the other hand, in our simulations, as pair arrivals are discrete and patients are heterogenous in their tissue-type compatibility probabilities, there could be potential welfare gains from the participation of compatible type $X - X$ pairs in exchange with incompatible $X - X$ pairs already in the queue.

D.2 Simulation Results

The simulation results for service rates regarding two-way exchange are slightly lower than or comparable to those of the calibrated-model predictions for the de-facto deceased-donor allocation policy. Service rates are reported in Table 12. The new regimes, incentivized exchange with compatible type $X-X$ pairs, fare better than the incentivized regimes without compatible-type-$X-X$-pair participation. When compared with Table 4 in Section 5, the corresponding percentages are slightly lower than calibrated-model results in all exchange regimes. The simulation and calibration results are similar to each other for only deceased-donor transplantation regime and deceased-donor/direct living-donor transplantation regime. This can be attributed to the fact that overdemanded pairs with high-CPRA patients do not necessarily participate in an exchange in the simulations while they do under the continuum-model assumptions.

The simulation results regarding two-and-three-way exchange are reported in Table 13 for service rates. We also plot the comparison of service rates for paired patients to receive a living-donor transplant between two-way exchange and two-and-three-way exchange in Figure 1 and for all transplants in Figure 2. In the first figure (as well as Tables 12 and 13), we observe that 67.7 percent of living donation candidates are served through two-and-three-way exchanges in addition

to direct donation when $X-X$ pairs are not incentivized. This rate is 66.5 percent under two-way exchange. Under $\rho = 30$ percent participation in incentivized exchange, these rates go up to 75.5 percent and 72.1 percent, respectively (resulting in a 3.4 percent difference). There is a further increase in the difference of the service rates at $\rho = 50$ percent. The marginal impact of three-way exchange technology slightly decreases at $\rho = 100$ percent. This is expected as most gains from exchange are utilized through higher incentivized participation rates. There is one contribution of three-way exchange that higher incentivized participation rates cannot compensate under two-way exchange: According to our model calibration, annually more $B-A$ pairs arrive than $A-B$ pairs do. Thus, all type $B-A$ pairs cannot be matched in our optimal two-way exchange policy. As ρ increases, all remaining $B-A$ pairs can be matched through three-way exchanges consisting of pairs of types $A-O$, $O-B$, $B-A$ or $AB-B$, $B-A$, $A-AB$ (see Roth, Sönmez, and Ünver, 2007 for details). The figure also shows that once all the remaining $B-A$ pairs can be matched through three-way exchanges, even if ρ increases further, the marginal gains from three-way exchange no longer increases. About 98.5 percent or more of all paired A, B, and AB patients are matched under two-and-three-way exchange policy when $\rho = 100$ percent (see the last row of Table 13), while this rate is lower for B under two-way exchange (around 93.9 percent).

When compatible pairs of types $X-X$ are also incentivized, we observe almost no difference under two-and-three-way exchange. However, compatible-$X-X$-pair participation has a much higher impact when three-way exchange technology is unavailable. For example, for $\rho = 20$ percent, 71.1 percent of all paired patients are matched when compatible $X-X$ pairs are incentivized. When they are not incentivized, the service rate is 70.3 percent. Differences stand for different ρ values. The reason for this disparity is the flexibility provided by three-way exchanges. An incompatible type $X-X$ pair can potentially be inserted in a two-way exchange that includes a blood-type X patient. For example, a two-way exchange of pair types $(X-Y, Y-X)$ can be extended to a three-way exchange as $(X-X, X-Y, Y-X)$.[38] Thus, such a couple of $X-Y$ and $Y-X$ pairs plays a role similar to a single compatible $X-X$ pair; they both help an incompatible $X-X$ pair to be matched through an exchange. Thus, the availability of three-way exchange almost perfectly substitutes for compatible-$X-X$-pair participation in matching incompatible pairs.

We also give some absolute numbers from our simulations that are multiplied by 20, the simulation scale parameter, to compare them with the continuum model's predictions. We observe that the simulations lead to an average of $4,994$ (with a standard error of 16.8) direct-living annual donor transplants compared to the real-life number of $4,979$, which is our calibration parameter for the continuum model. Two-way exchange and two-and-three-way exchange add, respectively, average 998 and $1,100$ transplants annually opposed to $1,135$ of the calibrated continuum model, which exclusively uses two-way exchange. At each $\Delta\rho = 10$ percent participation increase in incentivized exchange, additional averages of 172/184 (depending on $X-X$ pairs are incentivized or not) and 188 annual transplants are conducted under two-way and two-and-three-way exchange simulations,

[38] Even when the patient of the type $X-Y$ pair is tissue-type incompatible with the donor of the type $Y-X$ pair and a two-way exchange is not feasible between these two pairs, the $X-X$ pair potentially can be inserted to create a three-way exchange benefitting three additional patients.

respectively. Recall that this number was 180 in the continuum model calibrations. Thus, while finite market simulations lead to slightly less regular exchange transplants due to the frictions in its more realistic discrete setup, incentivized exchange protocols seem to overcome these frictions such that simulations lead to at least as well or more added transplants than the continuum model's predictions. We explained its reasons in Section 5.2 in the main text before.

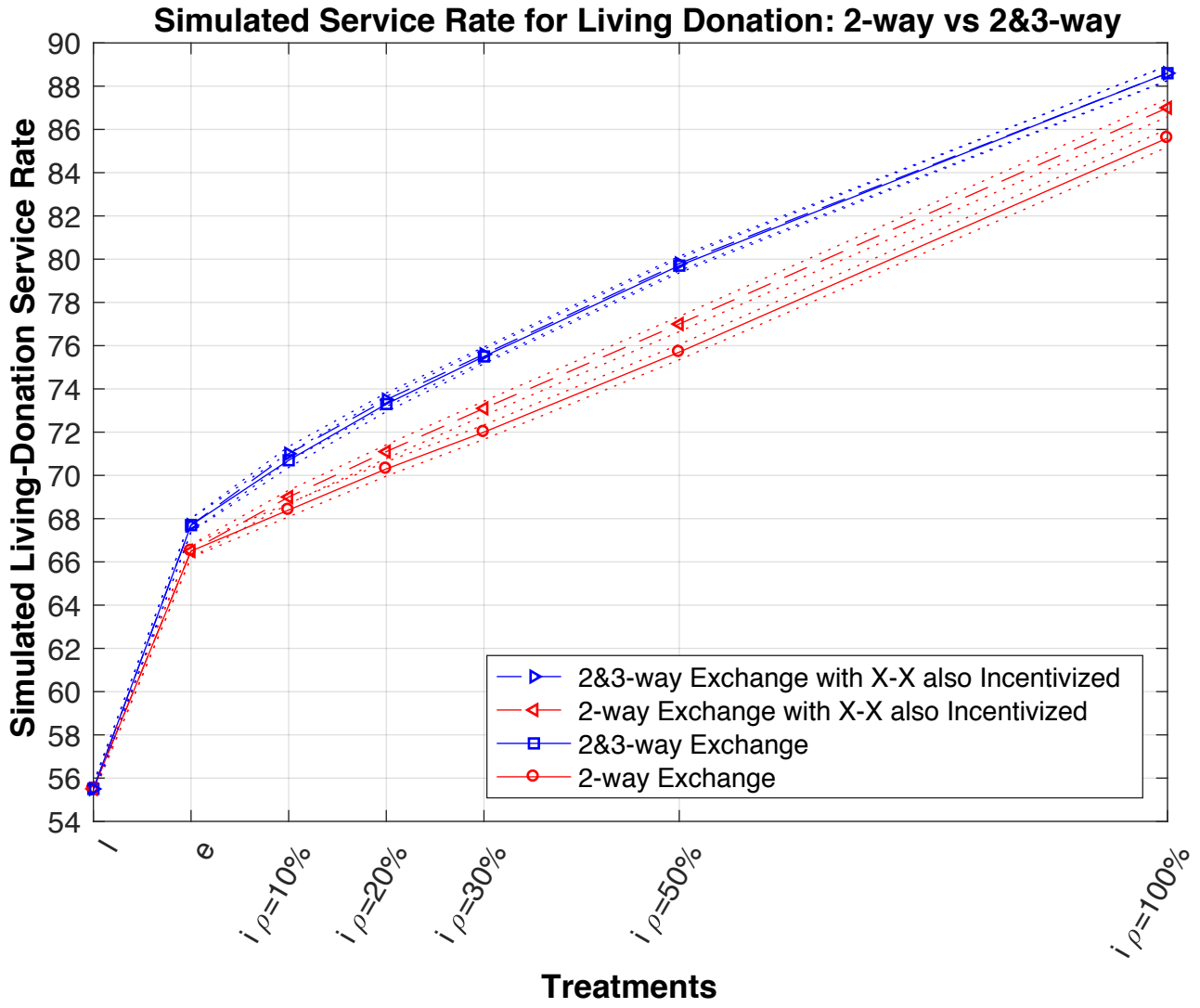

Figure 1: Simulation results for service rates for paired patients to receive living-donor transplants under two-way vs two-and-three-way exchange. Dotted lines are 95% confidence intervals for the averages.

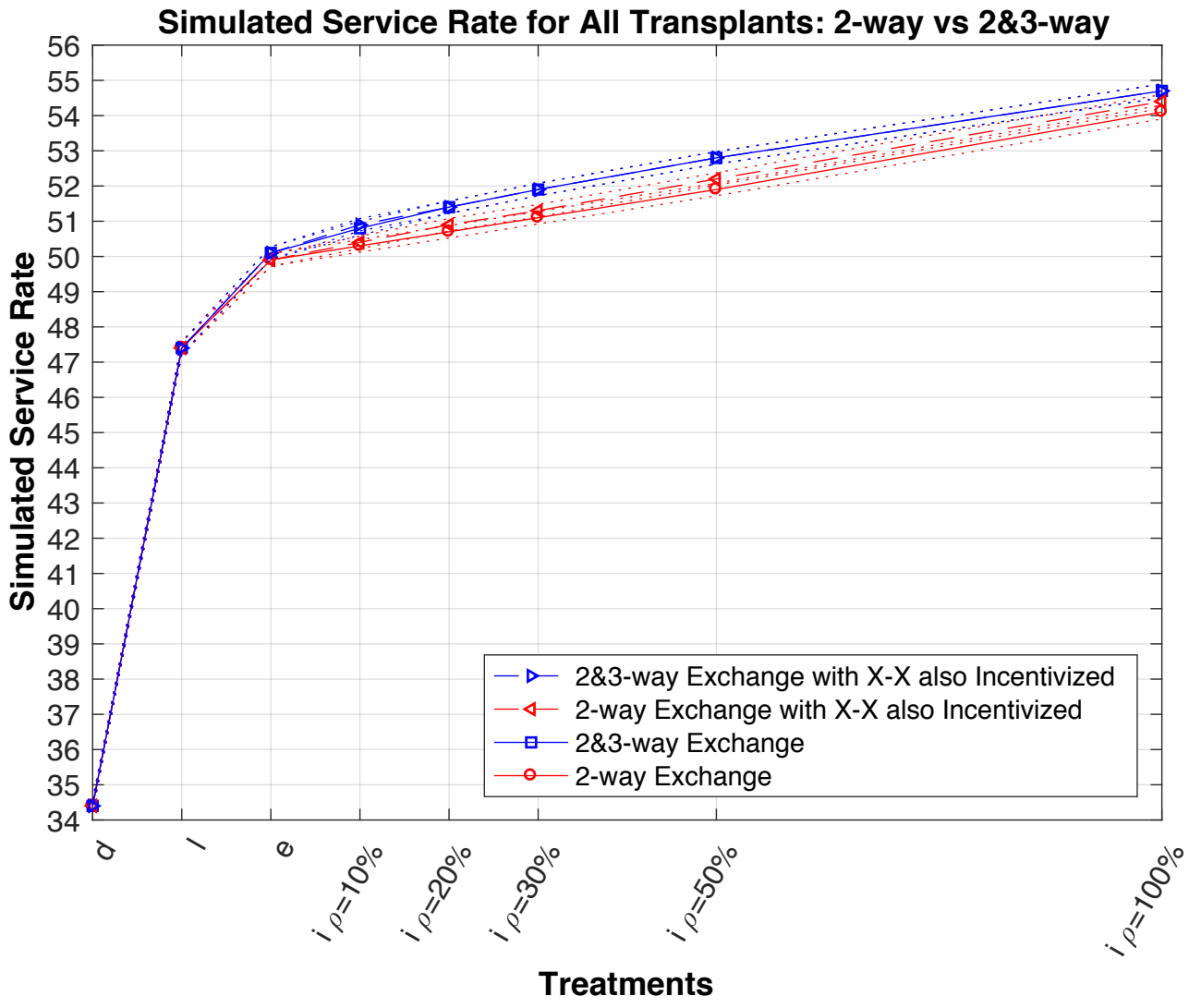

Figure 2: Simulation results for service rates for all transplants under two-way vs two-and-three-way exchange. Dotted lines are 95% confidence intervals for the averages.

Simulation Results: Service Rate for Transplantation in % under De-facto Deceased-Donor Allocation and 2-way Exchange															
Living-Donor Trans.					**Deceased-Donor Trans.**					**All Transplants**					
O	*A*	*B*	*AB*	All	*O*	*A*	*B*	*AB*	All	*O*	*A*	*B*	*AB*	All	
Deceased-Donor Transplantation Only (d)															
					32.5	37.6	30.8	46.5	34.4	32.5	37.6	30.8	46.5	34.4	
					(1.2)	(1.6)	(2.1)	(5.0)	(0.8)	(1.2)	(1.6)	(2.1)	(5.0)	(0.8)	
Deceased-/Direct Living-Donor Transplantation (l)															
42.7	78.0	54.2	92.9	55.5	36.0	43.4	34.4	52.7	38.7	44.0	53.0	42.9	59.6	47.4	
(1.9)	(1.9)	(3.7)	(5.0)	(1.3)	(1.3)	(1.9)	(2.3)	(5.6)	(0.9)	(1.3)	(1.8)	(2.2)	(5.2)	(0.9)	
Deceased-/Direct Living-Donor Transplantation & Regular Exchange (e)															
46.8	94.7	82.8	99.0	66.5	36.4	44.8	36.5	53.4	39.7	45.1	56.0	48.9	60.7	49.9	
(2.0)	(1.2)	(5.3)	(1.9)	(1.5)	(1.4)	(2.1)	(2.4)	(6.0)	(0.9)	(1.4)	(1.9)	(2.2)	(5.6)	(0.9)	
Deceased-/Direct Living-Donor Transplantation, Regular & $\rho = 10\%$-Incentivized Exchange (i $\rho = 10\%$)															
49.9	95.1	83.3	99.2	68.4	36.7	44.9	36.5	53.1	39.9	45.9	56.1	49.0	60.5	50.3	
(2.1)	(1.2)	(5.3)	(2.0)	(1.6)	(1.4)	(2.0)	(2.6)	(5.7)	(0.9)	(1.3)	(1.9)	(2.5)	(5.3)	(0.9)	
Deceased-/Direct Living-Donor Transplantation, Regular & $\rho = 20\%$-Incentivized Exchange (i $\rho = 20\%$)															
52.9	95.5	83.8	99.1	70.3	36.9	44.9	36.6	53.2	40.1	46.7	56.2	49.1	60.5	50.7	
(2.2)	(1.3)	(5.1)	(2.1)	(1.7)	(1.3)	(2.1)	(2.4)	(5.7)	(0.9)	(1.3)	(1.9)	(2.3)	(5.3)	(0.9)	
Deceased-/Direct Living-Donor Transplantation, Regular & $\rho = 30\%$-Incentivized Exchange (i $\rho = 30\%$)															
55.5	96.2	84.7	99.2	72.0	37.2	45.0	36.6	53.4	40.2	47.3	56.4	49.2	60.8	51.1	
(2.2)	(1.3)	(5.0)	(1.8)	(1.7)	(1.4)	(2.0)	(2.6)	(5.7)	(0.9)	(1.3)	(1.9)	(2.4)	(5.2)	(0.9)	
Deceased-/Direct Living-Donor Transplantation, Regular & $\rho = 50\%$-Incentivized Exchange (i $\rho = 50\%$)															
60.8	97.3	87.6	99.1	75.7	37.7	45.1	36.9	53.2	40.6	48.6	56.6	50.0	60.5	51.9	
(2.2)	(1.3)	(5.2)	(2.7)	(1.8)	(1.4)	(2.0)	(2.6)	(5.8)	(0.9)	(1.3)	(1.8)	(2.4)	(5.3)	(0.9)	
Deceased-/Direct Living-Donor Transplantation, Regular & $\rho = 100\%$-Incentivized Exchange (i $\rho = 100\%$)															
76.2	98.8	94.3	99.2	85.6	39.3	45.2	37.5	53.4	41.5	52.6	56.8	51.3	60.8	54.1	
(2.8)	(1.3)	(5.2)	(2.2)	(2.1)	(1.4)	(2.0)	(2.6)	(6.1)	(1.0)	(1.3)	(1.8)	(2.4)	(5.6)	(1.0)	
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 10\%$-Incentivized Exchange with $X-X$ (i $\rho = 10\%$)															
50.6	95.7	83.7	99.3	69.0	36.7	44.9	36.7	53.1	39.9	46.1	56.2	49.2	60.5	50.4	
(2.2)	(1.2)	(5.2)	(1.8)	(1.6)	(1.4)	(2.0)	(2.5)	(5.7)	(0.9)	(1.4)	(1.8)	(2.4)	(5.3)	(0.9)	
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 20\%$-Incentivized Exchange with $X-X$ (i $\rho = 20\%$)															
54.0	96.1	84.3	99.2	71.1	37.1	44.9	36.6	53.4	40.1	47.0	56.3	49.2	60.7	50.9	
(2.2)	(1.3)	(5.3)	(2.1)	(1.6)	(1.4)	(2.1)	(2.6)	(6.1)	(0.9)	(1.3)	(1.9)	(2.4)	(5.5)	(0.9)	
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 30\%$-Incentivized Exchange with $X-X$ (i $\rho = 30\%$)															
56.9	96.7	85.7	99.3	73.1	37.3	45.0	36.8	53.6	40.3	47.7	56.4	49.5	60.9	51.3	
(2.2)	(1.2)	(5.3)	(1.8)	(1.7)	(1.4)	(2.0)	(2.5)	(5.9)	(0.9)	(1.3)	(1.8)	(2.4)	(5.4)	(0.9)	
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 50\%$-Incentivized Exchange with $X-X$ (i $\rho = 50\%$)															
62.8	97.6	88.6	99.3	77.0	37.9	45.1	36.9	53.5	40.7	49.2	56.6	50.1	60.8	52.2	
(2.2)	(1.3)	(5.3)	(2.3)	(1.8)	(1.4)	(2.0)	(2.7)	(6.1)	(0.9)	(1.3)	(1.9)	(2.5)	(5.7)	(0.9)	
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 100\%$-Incentivized Exchange with $X-X$ (i $\rho = 100\%$)															
78.7	99.0	93.9	99.2	87.0	39.6	45.3	37.4	53.2	41.7	53.2	56.9	51.2	60.5	54.4	
(2.7)	(1.4)	(5.2)	(2.1)	(2.1)	(1.5)	(2.0)	(2.5)	(5.9)	(1.0)	(1.3)	(1.8)	(2.4)	(5.4)	(1.0)	

Table 12: Simulation results for service rates for paired patients to receive living-donor transplants, service rates for deceased-donor-queue participants, and overall service rates for patients to receive any kind of transplant (all measured in %) under de-facto deceased-donor allocation policy and two-way exchange when compatible $X-X$ pairs are not incentivized (middle four rows) and incentivized (last four rows).

Simulation Results: Service Rate for Transplantation in % under De-facto Deceased-Donor Allocation and 2&3-way Exchange															
Living-Donor Trans.					**Deceased-Donor Trans.**					**All Transplants**					
O	*A*	*B*	*AB*	All	*O*	*A*	*B*	*AB*	All	*O*	*A*	*B*	*AB*	All	
Deceased-Donor Transplantation Only (d)															
					32.5	37.6	30.8	46.5	34.4	32.5	37.6	30.8	46.5	34.4	
					(1.2)	(1.6)	(2.1)	(5.0)	(0.8)	(1.2)	(1.6)	(2.1)	(5.0)	(0.8)	
Deceased-/Direct Living-Donor Transplantation (l)															
42.7	78.0	54.2	92.9	55.5	36.0	43.4	34.4	52.7	38.7	44.0	53.0	42.9	59.6	47.4	
(1.9)	(1.9)	(3.7)	(5.0)	(1.3)	(1.3)	(1.9)	(2.3)	(5.6)	(0.9)	(1.3)	(1.8)	(2.2)	(5.2)	(0.9)	
Deceased-/Direct Living-Donor Transplantation & Regular Exchange (e)															
47.3	96.1	86.3	99.5	67.7	36.4	44.9	36.8	53.3	39.8	45.2	56.3	49.6	60.7	50.1	
(2.1)	(1.1)	(5.2)	(1.5)	(1.6)	(1.4)	(2.0)	(2.6)	(6.3)	(0.9)	(1.4)	(1.9)	(2.4)	(5.8)	(0.9)	
Deceased-/Direct Living-Donor Transplantation, Regular & $\rho = 10\%$-Incentivized Exchange (i $\rho = 10\%$)															
50.8	96.9	92.4	99.6	70.7	36.7	45.0	37.5	53.4	40.1	46.1	56.4	51.1	60.7	50.8	
(2.1)	(1.2)	(6.0)	(1.3)	(1.7)	(1.4)	(2.0)	(2.7)	(5.8)	(0.9)	(1.4)	(1.8)	(2.6)	(5.3)	(1.0)	
Deceased-/Direct Living-Donor Transplantation, Regular & $\rho = 20\%$-Incentivized Exchange (i $\rho = 20\%$)															
54.4	97.4	95.8	99.8	73.3	37.1	45.1	37.5	53.5	40.3	47.1	56.5	51.5	60.9	51.4	
(2.3)	(1.3)	(4.9)	(0.9)	(1.8)	(1.4)	(2.0)	(2.7)	(6.0)	(0.9)	(1.4)	(1.9)	(2.5)	(5.5)	(0.9)	
Deceased-/Direct Living-Donor Transplantation, Regular & $\rho = 30\%$-Incentivized Exchange (i $\rho = 30\%$)															
57.9	97.9	96.2	99.7	75.5	37.4	45.1	37.7	53.6	40.6	47.9	56.6	51.8	60.9	51.9	
(2.3)	(1.4)	(3.2)	(1.3)	(1.7)	(1.4)	(2.1)	(2.6)	(5.9)	(0.9)	(1.4)	(1.9)	(2.4)	(5.4)	(0.9)	
Deceased-/Direct Living-Donor Transplantation, Regular & $\rho = 50\%$-Incentivized Exchange (i $\rho = 50\%$)															
64.7	98.7	97.6	99.5	79.7	38.1	45.2	37.7	53.4	41.0	49.7	56.8	51.9	60.8	52.8	
(2.3)	(1.3)	(2.5)	(1.4)	(1.7)	(1.4)	(2.0)	(2.8)	(6.1)	(0.9)	(1.3)	(1.8)	(2.5)	(5.6)	(0.9)	
Deceased-/Direct Living-Donor Transplantation, Regular & $\rho = 100\%$-Incentivized Exchange (i $\rho = 100\%$)															
80.0	99.7	98.9	99.6	88.6	39.7	45.3	37.9	53.4	41.8	53.5	57.0	52.2	60.7	54.7	
(2.9)	(0.7)	(2.4)	(1.5)	(1.8)	(1.5)	(2.0)	(2.6)	(5.9)	(1.0)	(1.4)	(1.8)	(2.4)	(5.4)	(1.0)	
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 10\%$-Incentivized Exchange with $X-X$ (i $\rho = 10\%$)															
51.0	96.9	93.1	99.7	71.0	36.8	45.0	37.3	53.3	40.1	46.2	56.5	51.0	60.7	50.9	
(2.1)	(1.1)	(5.7)	(1.4)	(1.7)	(1.3)	(2.0)	(2.6)	(5.7)	(0.9)	(1.3)	(1.9)	(2.5)	(5.2)	(0.9)	
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 20\%$-Incentivized Exchange with $X-X$ (i $\rho = 20\%$)															
54.7	97.4	95.9	99.8	73.5	37.1	45.1	37.6	53.5	40.4	47.1	56.6	51.6	60.9	51.4	
(2.1)	(1.2)	(3.9)	(1.1)	(1.6)	(1.3)	(2.1)	(2.6)	(5.8)	(0.9)	(1.3)	(1.9)	(2.5)	(5.3)	(0.9)	
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 30\%$-Incentivized Exchange with $X-X$ (i $\rho = 30\%$)															
58.1	97.9	96.5	99.7	75.6	37.5	45.1	37.6	53.5	40.6	48.0	56.6	51.7	60.9	51.9	
(2.3)	(1.3)	(2.6)	(1.3)	(1.7)	(1.4)	(2.1)	(2.6)	(6.0)	(0.9)	(1.4)	(1.9)	(2.4)	(5.5)	(0.9)	
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 50\%$-Incentivized Exchange with $X-X$ (i $\rho = 50\%$)															
64.8	98.8	97.8	99.6	79.8	38.1	45.2	37.7	53.4	41.0	49.7	56.8	52.0	60.7	52.8	
(2.4)	(1.3)	(2.0)	(1.5)	(1.7)	(1.4)	(2.1)	(2.7)	(6.0)	(0.9)	(1.3)	(1.9)	(2.5)	(5.5)	(0.9)	
Deceased-/Direct Living-Donor Trans., Regular & $\rho = 100\%$-Incentivized Exchange with $X-X$ (i $\rho = 100\%$)															
80.1	99.6	98.7	99.6	88.6	39.8	45.3	37.9	53.3	41.9	53.5	57.0	52.2	60.7	54.7	
(2.8)	(0.7)	(2.7)	(1.7)	(1.7)	(1.5)	(2.0)	(2.6)	(5.8)	(1.0)	(1.5)	(1.8)	(2.4)	(5.3)	(1.0)	

Table 13: Simulation results for service rates for paired patients to receive living-donor transplants, service rates for deceased-donor-queue participants, and overall service rates for patients to receive any kind of transplant (all measured in %) under de-facto deceased-donor allocation policy and two-and-three-way exchange when compatible $X-X$ pairs are not incentivized (middle four rows) and incentivized (last four rows).

Appendix E Remaining Stress Tests

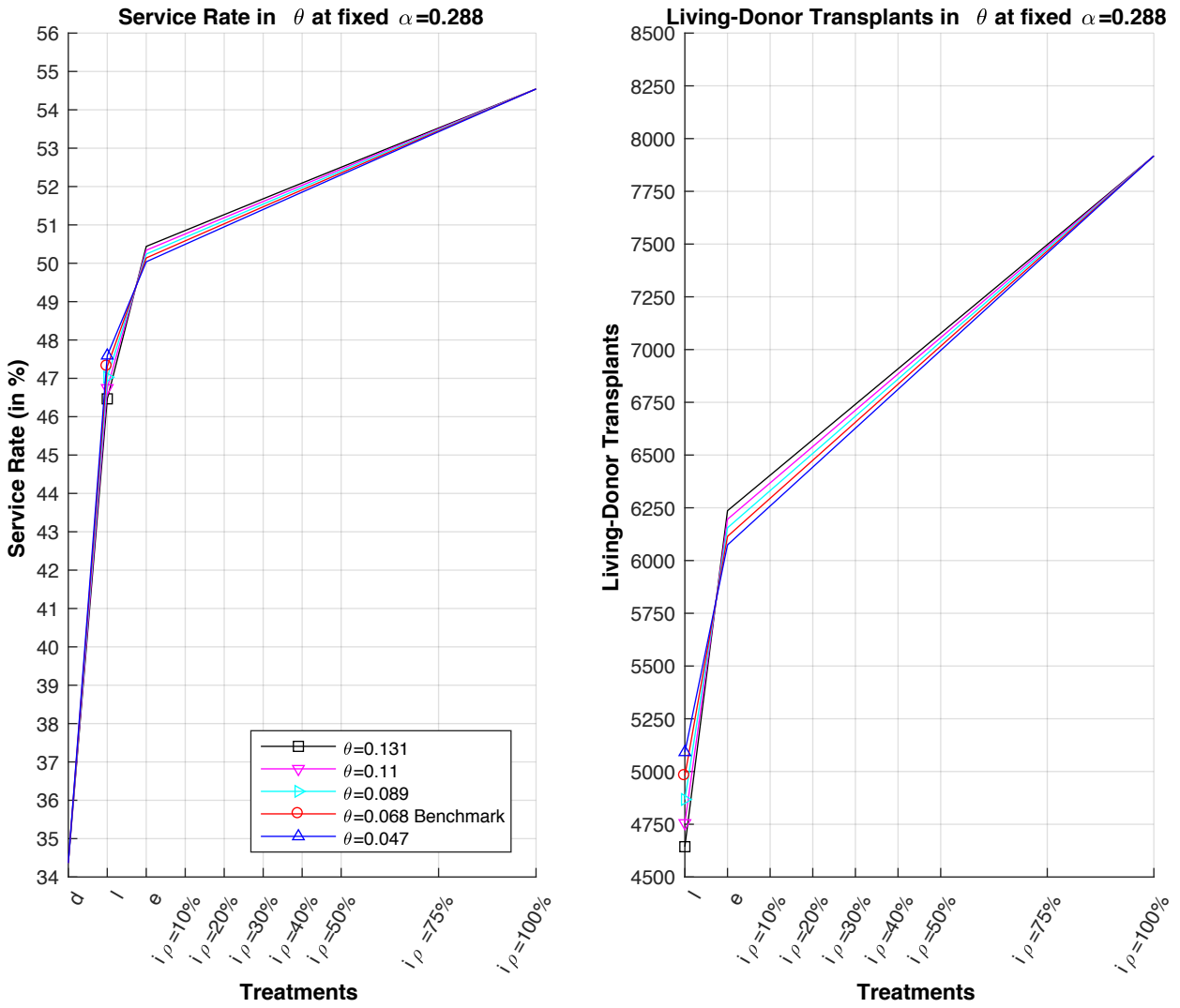

Figure 3: Stress tests for the numerical predictions of the model discussed in Section 5.2: Overall service rate and service rate of paired patients to receive a living-donor transplant in changing θ assuming (α_X) is fixed at its benchmark level with the average 0.288.

We report the results of the remaining stress test discussed in Section 5.2. Note that Figure 3 is already discussed in detail in Section 5.2.

Next we assume that θ is not precisely known. For each given θ, we find what levels of (α_X) will be necessary to support the observed direct living-donor transplant numbers $(\boldsymbol{\lambda}_X)$ in the data given in Table 6. The set of corresponding (θ, α) pairs is $\{(0.047, 0.282), (0.068, 0.288), (0.089, 0.295), (0.11, 0.301), (0.131, 0.309)\}$, where α is the mean probability of a random patient to have a paired donor for the calibrated (α_X) values.

Increasing θ means that a lower number of patients can receive direct transplants from their own donors. Since $(\boldsymbol{\lambda}_X)$ are kept constant, an increasing θ corresponds to higher (α_X) values. As

a result, the service rates and number of living transplants increase uniformly for all ρ values with increasing θ and (α_X) (see Figure 4). Each 0.021 probability increase in θ that corresponds to 0.007 increase in pairing rate leads to 180 additional transplants per year, 3.6 percent of direct living-donor transplants. The comparative static results regarding changes in ρ that we reported in Section 5 remain intact for different (θ, α) pairs.

Thus, changes in θ accompanied with induced changes in (α_X) to keep the number of direct living-donor transplants constant at its observed level has no effect on the impact of incentivized exchange, it only effects the number of patients that benefit from regular exchange.

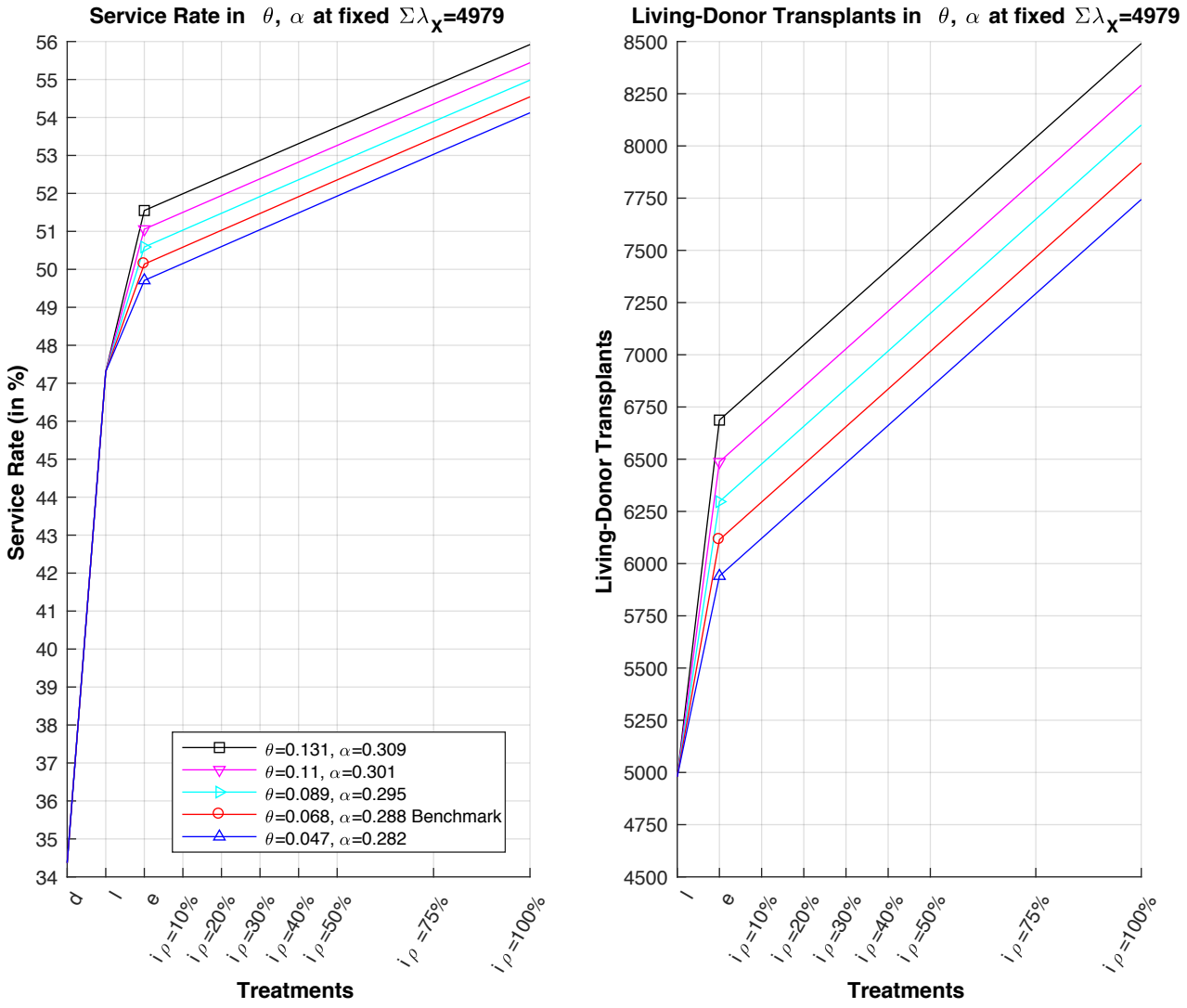

Figure 4: Stress tests for the numerical predictions of the model discussed in Appendix E: Stress tests of total service rate and total living-donor transplants in changing θ and (α_X) assuming $(\boldsymbol{\lambda}_X)$ is fixed at its benchmark level with the total $\sum \boldsymbol{\lambda}_X = 4979$.

Appendix F Perfect Matching with (Heterogenous) Tissue-Type Incompatibilities

In this appendix, we study the limit assumptions on the patient types under which different populations of pairs can be matched or patients can be assigned deceased-donor kidneys. The lemmas that we establish below are used in all results regarding steady states of the transplantation policies.

F.1 Matching Deceased-Donor Kidneys

We first consider the case when deceased-donor kidneys are matched with patients. We make the following regularity assumption on the frequency and incompatibility probability of patient types.

Assumption 4 *For every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$, such that for every $k > k_0$ and $l \leq k$ and every permutation σ of patient types,*

$$(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq 1 - \prod_{i=1}^{l} \theta_{\sigma(i),k}.$$

When $\epsilon \to 0$, the regularity assumption can be rewritten as $\sum_{i=l+1}^{k} m_{\sigma(i),k} \geq \prod_{i=1}^{l} \theta_{\sigma(i),k}$. It implies that if you take a set of patients and a set of kidneys with the same measure, then for any set of patient types the measure of patients with those types is greater than or equal to the measure of the set of kidneys that are tissue-type incompatible with all the other patient types.

Under this assumption, we get the following result.

Lemma 3 *Suppose Assumption 4 holds. Consider a measurable set of patients and deceased-donor kidneys that are blood-type compatible with all the patients such that both sets have the same measure. Suppose these sets are formed randomly using the governing population distributions. Then, as the number of patient types k goes to infinity, almost every patient can be matched with a compatible kidney.*

Proof. Without loss of generality, consider the case when the measures of the two sets are the same and equal to one. Fix a small $\epsilon > 0$. By Assumption 4, there exists k_0 such that, for every $k > k_0$, $l \leq k$, and permutation σ, $\frac{1-\prod_{i=1}^{l} \theta_{\sigma(i),k}}{\sum_{i=1}^{l} m_{\sigma(i),k}} \geq 1-\epsilon$. Consider any such k.

We use Gale's Supply-Demand Theorem (Gale, 1957) to show that $1-\epsilon$ measure of the kidneys can be matched with compatible patients. Consider a random measurable subset of patients with measure $1-\epsilon$. Since the subset is chosen randomly, the compatibility of patients with the kidneys can still be formed randomly using the governing population. We need to show that for any subset of patients, the measure of kidneys that are compatible with at least one patient is weakly greater than the measure of patients. Without loss of generality, instead of considering any set of patients we can consider the set of all patients who have types from any given set. Let the set of patient types be $\{\sigma(1), \ldots, \sigma(l)\}$.

The measure of patients that have a type in this set is $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}$. The measure of kidneys that are incompatible with all such types is $\prod_{i=1}^{l} \theta_{\sigma(i),k}$ because the measure of kidneys is

one. Therefore, the measure of kidneys that are compatible with at least one patient in the set is $1 - \prod_{i=1}^{n} \theta_{\sigma(i),k}$. The desired inequality holds by Assumption 4. The claim of the lemma follows by taking the limit as $\epsilon \to 0$ and $k \to \infty$. ■

F.2 Matching Type $A-B$ Pairs with Type $B-A$ Pairs

We next consider the case when we match reciprocal pairs, $A-B$ with $B-A$. For any such pair, tissue-type compatibility is not known because the pair is blood-type incompatible. Therefore, for any such pair, tissue-type incompatibility is determined randomly as in the overall population.

We make the following assumption on how the market grows, which guarantees that we can match almost every patient in two measurable sets of $A-B$ pairs and $B-A$ pairs that have the same measure.

Assumption 5 *For every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$, $l \leq k$, and every permutation σ of patient types,*

$$(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq \sum_{i=1}^{k} m_{\sigma(i),k}[1 - \prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))].$$

Consider two measurable sets of $A-B$ and $B-A$ pairs with the same measure. As $\epsilon \to 0$, the assumption guarantees that for any measurable set of reciprocal-type pairs, say $B-A$, the measure of this set is smaller than the measure of $A-B$ pairs that are compatible with at least one $B-A$ pair in this set.

Lemma 4 *Suppose Assumption 5 holds. Consider two measurable sets of $A-B$ and $B-A$ pairs that have the same measure. Suppose these sets are formed randomly using the governing population distributions. Then, as the number of patient types k goes to infinity, almost every pair can be matched with a compatible pair.*

Proof. Without loss of generality, consider the case when the measures of the two sets are the same and equal to one. Fix a small $\epsilon > 0$. By Assumption 5, there exists k_0 such that, for every $k > k_0$, $l \leq k$, and permutation σ, $\frac{\sum_{i=1}^{k} m_{\sigma(i),k}[1-\prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]}{\sum_{i=1}^{l} m_{\sigma(i),k}} \geq 1-\epsilon$. Consider any such k.

Like before, we use Gale's Supply-Demand Theorem (Gale, 1957) to show that $1-\epsilon$ measure of the $B-A$ pairs can be matched with compatible $A-B$ pairs. Consider a random measurable subset of $B-A$ pairs with measure $1-\epsilon$. Since the subset is chosen randomly, the compatibility of donors with patients can still be formed randomly using the governing population. We need to show that for any subset of $B-A$ pairs, the measure of $A-B$ pairs who are compatible with at least one $B-A$ pair in the chosen set is weakly greater than the measure of the chosen set of $B-A$ pairs. Without loss of generality, instead of considering any set of $B-A$ pairs, we can consider the set of all $B-A$ pairs with patients that have types from any given set. Let the set of patient types be $\{\sigma(1), \ldots, \sigma(l)\}$.

The measure of $B-A$ pairs with patients that have a type in this set is $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}$. The measure of $A-B$ pairs with patient type $\sigma(i)$ who are incompatible with all such pairs is $m_{\sigma(i),k}\prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))$. Therefore, the measure of $A-B$ pairs with patient type

$\sigma(i)$ who are compatible with at least one $B-A$ pair from the chosen set is $m_{\sigma(i),k}[1-\prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]$. Hence, the measure of $A-B$ pairs that are compatible with at least one $B-A$ pair in the chosen set is $\sum_{i=1}^{k} m_{\sigma(i),k}[1-\prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]$. This sum is greater than the measure of chosen $B-A$ pairs, $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}$, by Assumption 5.

Therefore, $1-\epsilon$ measure of $B-A$ pairs can be matched with compatible $A-B$ pairs. The lemma follows by taking $\epsilon \to 0$ and $k \to \infty$. ■

F.3 Matching Overdemanded-Type Pairs Except $A-B$ Pairs with Underdemanded-Type Pairs Except $B-A$ Pairs

We next consider the case when we match overdemanded-type pairs except $A-B$ pairs with underdemanded-type pairs except $B-A$ pairs. In the rest of this subsection, when we mention overdemanded-type pairs we exclude $A-B$ pairs, and similarly, when we mention underdemanded-type pairs we exclude $B-A$ pairs.

We make the following assumption on the frequency and incompatibility probability of patient types.

Assumption 6 *For every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$, $l \leq k$, $0 \leq \rho \leq 1$, and every permutation σ of patient types,*

$$(1-\epsilon)\sum_{i=1}^{l}\frac{m_{\sigma(i),k}(\theta_{\sigma(i),k}+\rho(1-\theta_{\sigma(i),k}))}{M} \leq \sum_{i=1}^{k} m_{\sigma(i),k}[1-\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))],$$

where $M=\sum_{i=1}^{k} m_{\sigma(i),k}(\theta_{\sigma(i),k}+\rho(1-\theta_{\sigma(i),k}))$.

For overdemanded-type pairs, only tissue-type-incompatible ones participate in the regular exchange. However, in the incentivized exchange, compatible pairs also participate. As a result, a fraction of the overdemanded pairs are compatible, while the rest are incompatible. Here, ρ is the participation rate of compatible pairs. The assumption guarantees that, for any set of overdemanded-type pairs, the set of underdemanded pairs that are compatible with at least one pair in the set has a greater measure as $\epsilon \to 0$.

Lemma 5 *Suppose Assumption 6 holds. Consider two measurable sets of overdemanded $X-Y$ pairs and underdemanded $Y-X$ pairs with the same measure. Suppose that a fraction of overdemanded $X-Y$ pairs are known to be tissue-type incompatible and the rest are known to be tissue-type compatible, but otherwise these sets are formed randomly using the governing population distributions. Then, as the number of patient types k goes to infinity, almost every pair can be matched with a compatible pair.*

Proof. Without loss of generality, consider the case when the measures of the two sets are the same and equal to one. Then, for underdemanded $Y-X$ pairs, $m_{i,k}$ measure of the patients have type i for every i. For overdemanded $X-Y$ pairs, some are known to be tissue-type compatible while others are tissue-type incompatible. The measure of compatible pairs is proportional to

$\rho m_{i,k}(1-\theta_{i,k})$ and the measure of incompatible pairs is proportional to $m_{i,k}\theta_{i,k}$. Therefore, the measure of overdemanded $X-Y$ pairs with patient type i is $\frac{m_{\sigma(i),k}(\theta_{\sigma(i),k}+\rho(1-\theta_{\sigma(i),k}))}{M}$ where $M = \sum_{i=1}^{k} m_{\sigma(i),k}(\theta_{\sigma(i),k}+\rho(1-\theta_{\sigma(i),k}))$.

Fix a small $\epsilon > 0$. Consider any k that satisfies Assumption 6. Like before, we use Gale's Supply-Demand Theorem (Gale, 1957) to show that $1-\epsilon$ measure of the overdemanded $X-Y$ pairs can be matched with compatible underdemanded $Y-X$ pairs. Consider a random measurable subset of overdemanded $X-Y$ pairs with measure $1-\epsilon$. Since the subset is chosen randomly, the compatibility of pairs can still be formed randomly using the governing population. We need to show that, for any subset of overdemanded $X-Y$ pairs, the measure of underdemanded $Y-X$ pairs who are compatible with at least one overdemanded $X-Y$ pair is weakly greater than the measure of overdemanded $X-Y$ pairs. In this calculation, we use a lower bound for the measure of such underdemanded $Y-X$ pairs by assuming that if their patient has type i, then they are incompatible with overdemanded $X-Y$ pairs with patient type i. Without loss of generality, instead of considering any set of overdemanded $X-Y$ pairs, we can consider the set of all overdemanded $X-Y$ pairs with patients who have tissue types from any given set. Let the set of patient types be $\{\sigma(1),\ldots,\sigma(l)\}$.

The measure of overdemanded $X-Y$ pairs with patients who have types in the set is $(1-\epsilon)\sum_{i=1}^{l}\frac{m_{\sigma(i),k}(\theta_{\sigma(i),k}+\rho(1-\theta_{\sigma(i),k}))}{M}$. The measure of underdemanded $Y-X$ pairs with patient type $\sigma(i)$ for $i \leq l$ who are incompatible with all such pairs is $m_{\sigma(i),k}\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))$. Note that we are assuming that these pairs are incompatible with overdemanded $X-Y$ pairs with patient of type $\sigma(i)$. On the other hand, if $i > l$, then the measure of underdemanded $Y-X$ pairs with patient type $\sigma(i)$ who are incompatible with all such pairs is $m_{\sigma(i),k}\prod_{j=1}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))$. Hence, the measure of underdemanded $Y-X$ pairs that are compatible with at least one overdemanded $X-Y$ pair in the chosen set is at least $\sum_{i=1}^{k} m_{\sigma(i),k}[1-\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]$. This sum is greater than the measure of chosen overdemanded $X-Y$ pairs Assumption 6.

The proof that $1-\epsilon$ measure of overdemanded $X-Y$ pairs can be matched follows. The lemma follows by taking $k\to\infty$ and $\epsilon\to 0$. ■

F.4 Matching Self-Demanded-Type Pairs

In this section, we consider the case when we match self-demanded type pairs. Fix any self-demanded-type pair $X-X$ for some blood type X. Any such pair in the exchange pool is tissue-type incompatible. We match these pairs with each other. Therefore, in contrast with the previous sections, this is a one-sided matching problem.

We make the following assumption to show that almost every pair can be matched in the limit.

Assumption 7 *For every $\epsilon > 0$, there exists $k_0 \in \mathbb{N}$ such that for every $k > k_0$, $l \leq k$, and every permutation σ of patient types,*

$$(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}\theta_{\sigma(i),k} \leq \sum_{i=1}^{k} m_{\sigma(i),k}\theta_{\sigma(i),k}[1-\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))].$$

Our next result shows that under this assumption almost all self-demanded pairs can be matched.

Lemma 6 *Suppose Assumption 7 holds. Consider a set of self-demanded-type pairs $X-X$ that are tissue-type incompatible. Assume that this set is formed randomly using the governing population distributions. Then, as the number of patient types k goes to infinity, almost every pair can be matched with a compatible pair.*

Proof. Since the pairs are tissue-type incompatible, but otherwise formed randomly using the governing population distributions, for each patient type i, the measure of pairs with patient type i is proportional to $m_i\theta_i$.

Fix a small $\epsilon > 0$. Consider any k that satisfies Assumption 7.

We use Gale's Supply-Demand Theorem (Gale, 1957) to show that $1-\epsilon$ fraction of the self-demanded $X-X$ pairs can be matched with compatible self-demanded $X-X$ pairs. To show this, we first construct a two-sided matching problem with these pairs. For any patient type i, we split the set of pairs with patient type i into two sets with equal measure. These sets are then added to different sides of the market. As a result, we get a two-sided matching problem where each side has $X-X$ pairs where those with patient type i have a measure proportional to $m_i\theta_i$. For ease of exposition, suppose that the measure is exactly $m_i\theta_i$.

Consider one side of the market. To apply Gale's Supply-Demand Theorem, take a random measurable subset of pairs on this side of the market that has measure $1-\epsilon$ fraction of all pairs on this side. Since the subset is chosen randomly, the compatibility of patients can still be formed randomly using the governing population. We need to show that for any subset of pairs, the measure of pairs on the other side of the market that are compatible with at least one pair in the set is weakly greater than the measure of chosen pairs. Without loss of generality, instead of considering any set of patient types, we can consider the set of all patients that have types from any given set. Let this set be $\{\sigma(1),\dots,\sigma(l)\}$.

The measure of the set of pairs that have patient types from this set is $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}\theta_{\sigma(i),k}$. The measure of pairs that have patient type $\sigma(i)$ on the other side that are incompatible with all such types is $m_{\sigma(i),k}\theta_{\sigma(i),k}\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))$. The measure of pairs that have patient type $\sigma(i)$ on the other side that are compatible with at least one type in the set is $m_{\sigma(i),k}\theta_{\sigma(i),k}[1-\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]$. Therefore, the measure of pairs on the other side that are compatible with at least one pair in the chosen set is $\sum_{i=1}^{k} m_{\sigma(i),k}\theta_{\sigma(i),k}[1-\prod_{j=1,j\neq i}^{l}(1-(1-\theta_{\sigma(j),k})(1-\theta_{\sigma(i),k}))]$. This sum is greater than the measure of pairs that are chosen, which is $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k}\theta_{\sigma(i),k}$ by Assumption 7.

Therefore, $1-\epsilon$ fraction of pairs on both sides of the market can be matched. As we take $\epsilon \to 0$ and $k \to \infty$, we establish the desired result that almost every pair is matched with a compatible pair. ■

F.5 Sufficient Limit Conditions

In the next lemma, we provide sufficient conditions under which all of the limit assumptions hold.

Lemma 7 *Suppose that $\theta_{i,k} = \theta < 1$ and $m_{i,k} \to 0$ for every $i \leq k$ as $k \to \infty$. Then Assumptions 4, 5, 6, and 7 hold.*

Proof. When $\theta_{i,k} = \theta$ for every $i \leq k$, Assumption 4 reduces to

$$(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq 1-\theta^l$$

under the same conditions as stated therein. Likewise, Assumption 5 reduces to

$$(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq 1-(1-(1-\theta)^2)^l,$$

and Assumptions 6 and 7 reduce to

$$(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq \sum_{i=1}^{l} m_{\sigma(i),k}[1-(1-(1-\theta)^2)^{l-1}] + \sum_{i=l+1}^{k} m_{\sigma(i),k}[1-(1-(1-\theta)^2)^l].$$

If we show that $(1-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq \sum_{i=1}^{l} m_{\sigma(i),k}[1-\beta^{l-1}] + \sum_{i=l+1}^{k} m_{\sigma(i),k}[1-\beta^l]$ for every $\beta < 1$ under the conditions stated in these assumptions, then we will be done. This inequality can be rewritten as

$$(\beta^{l-1}-\epsilon)\sum_{i=1}^{l} m_{\sigma(i),k} \leq \sum_{i=l+1}^{k} m_{\sigma(i),k}[1-\beta^l]. \tag{17}$$

For a fixed ϵ such that $1 > \epsilon > 0$, there exists a natural number n such that $\beta^{n-1} \geq \epsilon > \beta^n$. Then Inequality 17 holds for $l > n$ for every k because the left side of the inequality is negative whereas the right side is positive. Furthermore, as $k \to \infty$ Inequality 17 holds also for every $l \leq n$ because $m_{i,k} \to 0$ for every i and n is a fixed natural number which does not depend on k. In this case, the left side converges to zero and the right side is always positive. ■