# Efficient and Incentive-Compatible Liver Exchange

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Arrow Lecture

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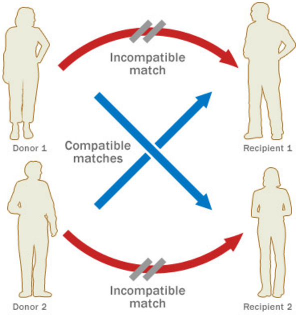
# Introduction

- Kidney Exchange became a mainstream transplantation modality within the last fifteen years.
- Annually, more than 700 patients in the US receive kidney transplants through donor exchange.
- In theory living-donor organ exchange can be utilized for any organ for which living donation is feasible.
- Liver is the second most transplanted organ following the kidney.
- Living donation of a lobe of liver is widespread, especially in Asia.

# Kidney Exchange




- Human organs cannot received or given in exchange for "valuable consideration" (US, NOTA 1984, WHO)
- However, living-donor kidney exchange is not considered as "valuable consideration" (US NOTA amendment, 2007)

# Literature

- **Kidney Exchange Literature**: Plenty...
- **Liver Exchange Literature**:
  - Hwang et al. [10] proposed the idea and documented the practice in Korea since 03
  - Chen et al. [10] documented the program in Hong Kong
  - Dickerson & Sandholm [14] asymptotic gains from liver+kidney exchange over isolated liver exchange and kidney exchange
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# Contribution

- We model liver exchange as a market design problem – different than kidney exchange due to size-compatibility requirement, and the availability of multiple transplant technologies.
- We find the structure of feasible 2-way exchanges and a sequential algorithm to find an efficient matching for two patient/donor sizes.
- The requirement of size compatibility induces an incentive problem for the pair/donor to donate
  - the larger/riskier/easier to match right lobe or
  - the smaller/safer/more difficult to match left lobe
- For any given number of patient/donor sizes, we propose a Pareto-efficient and incentive-compatible mechanism that elicits willingness to donate the right lobe truthfully.
- We introduce a new class of exchange mechanisms for vector-partial-order-induced weak preferences.

# Institutions: Living-Donor Liver Transplantation

- Living-donor liver transplantation is the norm in Asian countries, where deceased-donor transplantation is much less common due to cultural reasons and legal non-recognition of brain death.

Annual liver transplant activity per million population

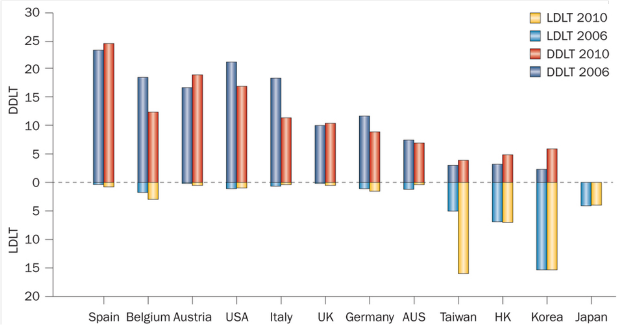
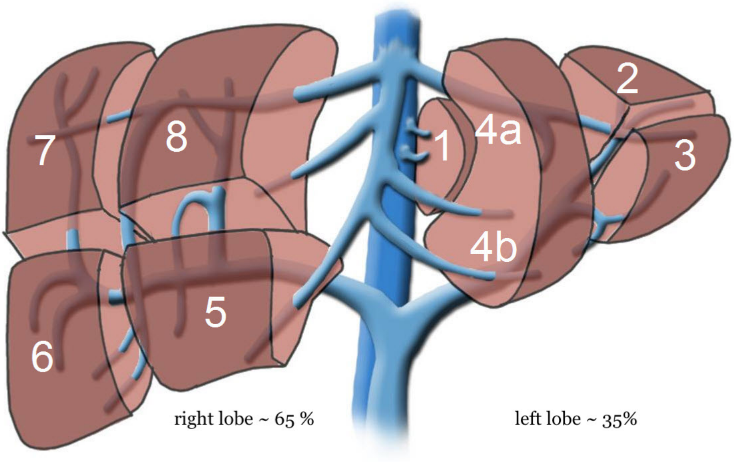


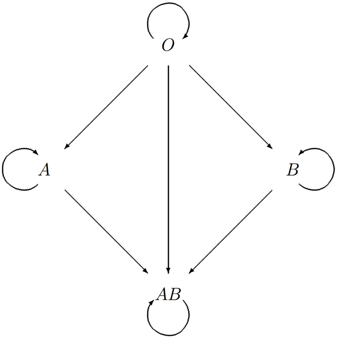

Figure from Chen et al Nature Reviews Gastroenterology & Hepatology 2013

# Medical Background: Lobar Liver Donation



# Medical Background: Compatibility

- As in kidney transplantation, blood-type compatibility is required.




- Different than kidney transplantation,
  - tissue-type compatibility is **not** required, but instead
  - size compatibility is required: A patient is in need of a graft that is at least 40% of the volume of his dysfunctional liver.

# Institutions: Right-Lobe Liver Transplantation

- Right-lobe transplant has been utilized for size compatibility despite its heightened donor mortality risk.
  - Patient needs at least 40% of his own liver size to survive.
  - Usually right lobe is $\sim$60-70%, left lobe is $\sim$30-40% of the liver.
  - In many occasions, size compatibility is only satisfied through right-lobe transplantation.

# Institutions: Living Donor Deaths

TABLE 1. Deaths of Living Donors

| Reference | Date | Location | Description |
|---|---|---|---|
| **Donor deaths "definitely" related to donor hepatectomy** | | | |
| 11 | 2003 | Japan | A mother in her late 40s donated a right lobe and died 9 months later from complications of hepatic failure. |
| 12 | 2002 | USA | A 57-year-old brother donated a right lobe and developed gastric gas gangrene and *Clostridium perfringens* infection 3 days after surgery and died. |
| 13 | 2005 | Brazil | A 31-year-old female right lobe donor of unknown relationship to the recipient died 7 days after surgery from a subarachnoid hemorrhage. |
| 14 | 2003 | India | A donor of unknown age and unknown relationship to the recipient donated an unknown lobe and died 10 days after surgery of unknown causes. |
| 15 | 2003 | India | A 52-year-old wife donated an unknown lobe and became comatose 48 hours after surgery from unknown causes and remains in chronic vegetative state. |
| 16-18 | 1993 | Germany | A 29-year-old mother donated a left lateral lobe and died of a pulmonary embolus 48 hours after surgery. |
| 18, 19 | 2000 | Germany | A 38-year-old father donated a right lobe, and 32 days after developing progressive hepatic failure, died during transplantation of acute cardiac failure. The cause of the donor's death was attributed to Berardinelli-Seip syndrome, a lipodystrophy syndrome characterized by loss of body fat, diabetes, hepatomegaly, and acanthosis nigricans. |
| 18, 20 | 2000 | France | A 32-year-old brother donated a right lobe and developed sepsis and multiple organ system failure 11 days after surgery and died of septic shock 3 days later. |
| 18 | 2000 | Europe | A 57-year-old wife donated a right lobe and died of sepsis and multiple organ system failure 21 days after surgery. |
| 21, 22 | 1999 | USA | A 41-year-old half-brother donated a right lobe and died of pancreatitis and sepsis 1 month later. |
| 22, 23 | 1997 | USA | A mother of unknown age donated an unknown lobe to a pediatric recipient and died 3 days after surgery of unknown causes. |
| 24 | 2005 | Asia | A 50-year-old mother donated a right hepatic lobe. She had no history of peptic ulcer disease and received a 2-week course of H2 antagonist. She died 10 weeks after surgery from an autopsy-proven duodenal ulcer with a duodenocaval fistula causing air embolism. |
| 25 | 2006 | Asia | A 39-year-old male "close relative" who donated an unknown lobe died of a myocardial infarction 4 days after donation. The patient reportedly had a preoperative electrocardiogram and treadmill test. |
| 26 | 2005 | Egypt | A brother of unknown age who donated a right lobe died of complications of sepsis from a bile leak 1 month after donation. |
| **Donor deaths "possibly" related to donor hepatectomy** | | | |
| 27 | 2005 | USA | A 35-year-old brother donated a right lobe and died of a self-induced drug overdose 23 months later. |
| 27 | 2005 | USA | A 50-year-old uncle donated a right lobe and died of a self-inflicted gunshot wound to the head 22 months after donation. |
| **Donor deaths "unlikely" to be related to donor hepatectomy** | | | |
| 28 | 2003 | Asia | A donor of unknown age and relationship to the recipient who donated an unknown lobe died of unknown causes during exercise 3 years after donation. |
| 27, 29 | 2002 | USA | A 35-year-old boyfriend donated a right lobe and died in a nonsuicidal occupational pedestrian-train accident 2 years after donation. A lone railroad car rolling at high speed struck and killed the donor while he was on duty at his job for the railroad. |
| 16 | 2003 | Germany | A 30-year-old father donated a left lateral segment and died of complications of amyotrophic lateral sclerosis 11 years after successful donation. |
| 30 | 2003 | Japan | A male donor in his 40s of unknown relationship to the recipient donated an unknown lobe died 10 years postoperatively after an apparently unrelated surgery. |

- Donor mortality rate is 5 times higher for right-lobe donation than left-lobe donation (0.5% to 0.1%).
- Other significant risks, the morbidity rate, also much higher under right lobe donation (28% to 7.5%).
- In 2001, a high profile death of a living right-lobe liver donor in the US decreased living donation not only for livers, but also for kidneys.
- About half of the living-donor liver transplantations are from right lobes.

# Institutions: Living-Donor Liver Exchange

- Liver exchange was first practiced in Korea, followed by Hong Kong and Turkey.
- Liver exchange can have two benefits:
  (1) It can increase the number of transplants.
  (2) It can increase donor safety through an increased share of left-lobe transplants.

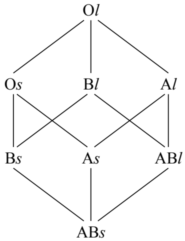
# Living-Donor Liver Exchange

- Liver exchange differs from kidney exchange in three key ways:
  (1) The lack of tissue-type incompatibility,
  (2) the presence of size incompatibility, and most notably
  (3) through two different transplant technologies: left-lobe transplantation and right-lobe transplantation.
- In the absence of size incompatibility the scope for liver exchange would be very limited: The only viable exchange would involve
  - a blood-type A patient with a blood-type B donor and
  - a blood-type B patient with a blood-type A donor.

# Liver Exchange Model: Two Patient/Donor Sizes

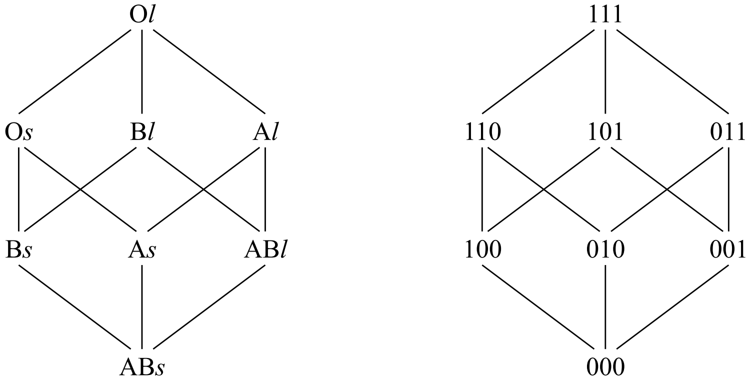
- $\underbrace{\{O, A, B, AB\}}_{\mathcal{B}} \times \underbrace{\{l, s\}}_{\mathcal{S}}$: Set of individual types
- Initial focus: Left-lobe-only liver transplants.
- Left-Lobe Compatibility: A patient can receive a left-lobe transplant from a donor if and only if
  (1) the patient is blood-type compatible with the donor, and
  (2) the donor is not smaller than the patient.

Liver Donation Partial Order $\trianglerighteq$ on $\mathcal{B} \times \mathcal{S}$



# An Equivalent Representation

- Consider the following two partially ordered sets:
  (1) The liver donation partial order $\trianglerighteq$ on $\mathcal{B} \times \mathcal{S}$, and
  (2) the standard partial order $\geq$ over the corners of the three-dimensional cube $\{0, 1\}^3$.



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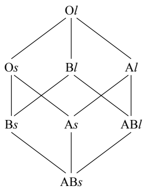
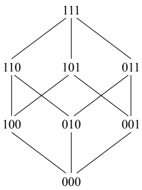



- Note that $(\mathcal{B} \times \mathcal{S}, \trianglerighteq)$ and $(\{0,1\}^3, \geq)$ are order isomorphic, where the order isomorphism associates each individual type $\tau \in \mathcal{B} \times \mathcal{S}$ with the following vector $X \in \{0,1\}^3$:

$$\begin{aligned} X_1 = 0 \quad &\Longleftrightarrow \quad \tau \text{ has the } A \text{ antigen} \\ X_2 = 0 \quad &\Longleftrightarrow \quad \tau \text{ has the } B \text{ antigen} \\ X_3 = 0 \quad &\Longleftrightarrow \quad \tau \text{ is small} \end{aligned}$$

- For notational convenience, we will work with the equivalent representation $(\{0,1\}^3, \geq)$.

- The type of a patient-donor pair is represented through the individual types of its patient and donor, respectively, as $X - Y \in \left(\{0,1\}^3\right)^2$.

**Definition**

A liver exchange problem is a list $\mathcal{E} = \{\mathcal{I}, \tau\}$ where $\mathcal{I} = \{1, 2, ..., I\}$ is a set of pairs, and for each $i \in \mathcal{I}$, $\tau(i) = X - Y$ is the type of pair $i$.

# Left-Lobe-Only Direct Transplant & 2-way Exchange

- A pair $i$ of type $X - Y$ is left-lobe compatible, if

$$Y \geq X$$

- A (left-lobe-only 2-way) liver exchange is feasible between a pair $i$ of type $X - Y$ and a pair $j$ of type $V - W$, if

$$Y \geq V \quad \text{and} \quad W \geq X$$

- A matching is a collection of mutually exclusive exchanges and direct transplants such that if a pair is left-lobe compatible, then it participates in a direct transplant.

# Value of a Pair-Type

- Value of a pair type $\underbrace{X_1X_2X_3}_{X} - \underbrace{Y_1Y_2Y_3}_{Y}$ is defined as

$$v(X - Y) = \sum_{k=1}^{3} (Y_k - X_k)$$

**Observation**

*In any liver exchange problem, the only types that could be part of an exchange are*

$$X - Y \in \left(\{0,1\}^3\right)^2 \text{ such that } X \not\geq Y \text{ and } Y \not\geq X.$$

*Therefore, only types of* ***values -1*****,** ***0*****,** *or* ***1*** *can be part of an exchange.*

# Waste of a 2-way Exchange

- Waste of an exchange between a pair of type $X - Y$ and a pair of type $V - W$ is defined as

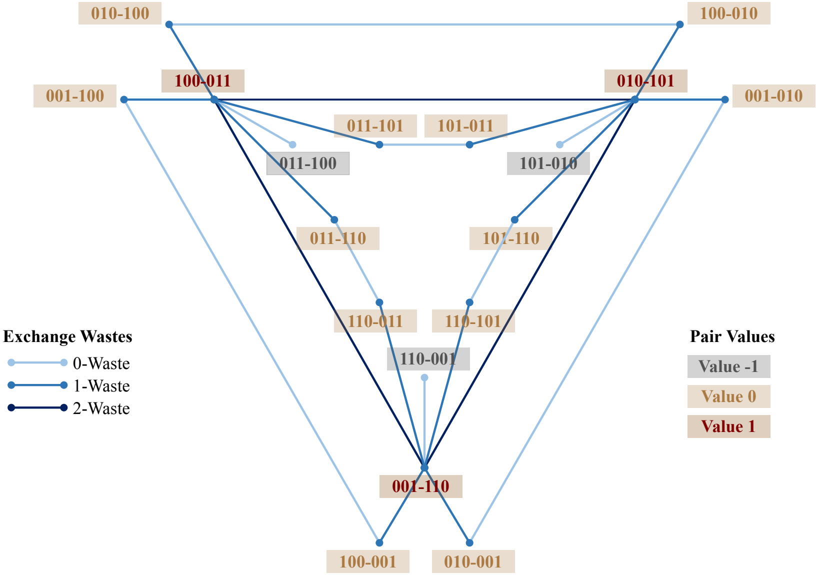
$$v(X - Y) + v(V - W)$$

- All feasible exchanges have non-negative waste.

**Observation**

*All feasible exchanges are either* ***0-waste****,* ***1-waste****, or* ***2-waste****.*

# Left-Lobe-Only 2-Way Exchange: Feasibility



# Two-Size Left-Lobe-Only Sequential Exchange Algorithm

Fix a priority order over pairs.

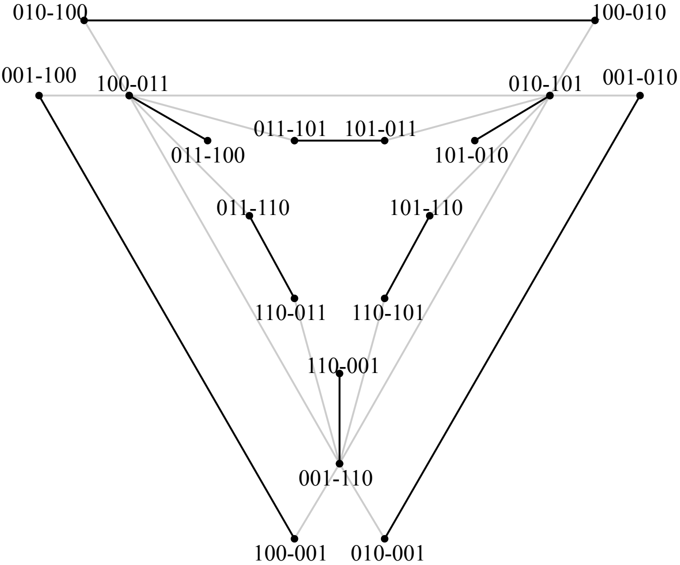
Step 0. Clear all feasible direct transplants.

Step 1. Clear **0-waste** exchanges following the given priority order.

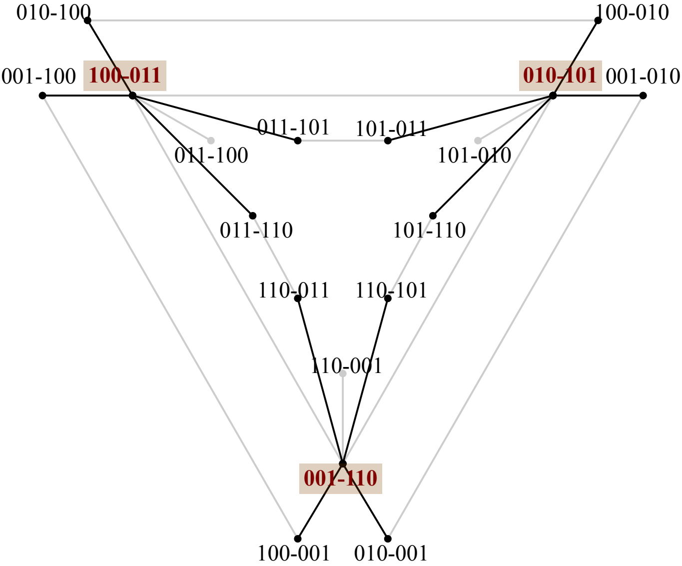
Step 2. Clear **1-waste** exchanges following the given priority order.

Step 3. Clear **2-waste** exchanges: Match the maximum number of value 1 types with each other, following the given priority order.

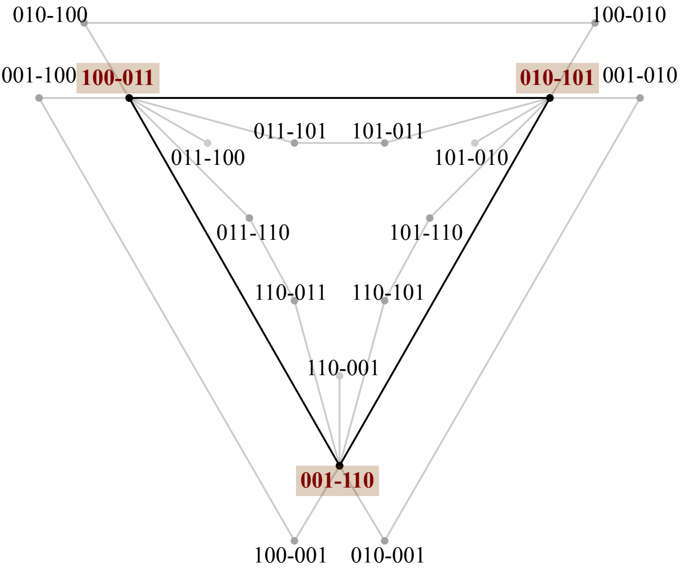
# Algorithm Step 1: Clear **0-waste** Exchanges



# Algorithm Step 2: Clear **1-waste** Exchanges



# Algorithm Step 3: Clear **2-waste** Exchanges



# Left-Lobe-Only 2-Way Exchange: Efficiency

**Theorem**

*For any liver exchange problem, the two-size left-lobe-only sequential exchange algorithm maximizes the number of left-lobe-only 2-way exchanges.*

# Right-Lobe Donation & Preferences

- Transplant Technologies:
  - Left-lobe transplant: A patient can receive a left-lobe transplant from a blood-type compatible donor who is at least as large.
  - Right-lobe transplant: A patient can receive a right-lobe transplant from a blood-type compatible donor of any size.
- Pair Preferences:
  - Left-lobe donation is preferred by any pair to right-lobe donation.
  - A willing $(w)$ pair prefers right-lobe donation to no-transplant.
  - An unwilling $(u)$ pair prefers no-transplant to right-lobe donation.

# Right-Lobe Donation & Preferences

Willing preferences $R_i^w$:

Left-Lobe Direct Transplant
Left-Lobe Exchange
Right-Lobe Direct Transplant
Right-Lobe Exchange
$\emptyset$

Unwilling preferences $R_i^u$:

Left-Lobe Direct Transplant
Left-Lobe Exchange
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# Right-Lobe Donation & Incentives

- Our focus is on individual rational exchanges:
  - A left-lobe compatible pair does not join in any exchange, but only in a left-lobe direct transplant.
  - A right-lobe-only compatible pair participates in an exchange only if its donor donates her left lobe; otherwise,
    - it participates in a right-lobe direct transplant if it is willing, and
    - it receives the no-transplant option if unwilling.
- Willingness (or equivalently preferences) of a pair is private information.
- We inspect direct revelation mechanisms to elicit willingness.
- Pairs may have incentives to hide their willingness.

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# Transition to Right-Lobe Donation: Transformation

- Fix a willingness profile $R = (R_i)_{i \in \mathcal{I}} \in \{R_i^u, R_i^w\}^{|\mathcal{I}|}$
- A pair of type $X_1X_2X_3 - Y_1Y_2 0w$ is treated as if it is of type $X_1X_2X_3 - Y_1Y_2 1$ when it donates a right lobe.

  We refer to this transition as a *transformation*.

## Lemma (Individually Rational Matchings)

*Given both transplant technologies, a pair type $X - Y$ belongs to one of the following seven disjoint groups, based on direct transplant and exchange options available to its members:*

0. $X > Y_1Y_21$: *cannot participate in an exchange or a direct transplant;*

I. $X \leq Y$: *participates in a direct left-lobe transplant;*

II. $Y_3 = 0$ & $X = Y_1Y_21$: *can only participate in a direct right-lobe transplant (if willing);*

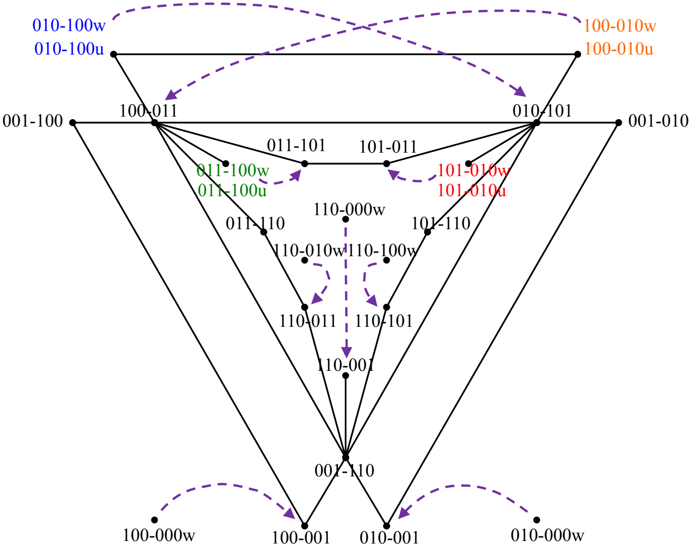
III. $Y_3 = 1$ & $X \not\geq Y$ & $X \not\leq Y$: *can only participate in exchange, and only by donating a left lobe;*

IV. $X_3 = 0, Y_3 = 0$ & $X > Y$: *can only participate in exchange, and only by donating a right lobe (if willing);*

V. $Y_3 = 0$ & $X \not\geq Y$ & $X \not\leq Y_1Y_21$ ($010 - 100$, $100 - 010$, $011 - 100$, $101 - 010$): *can only participate in exchange, either by donating a left lobe or a right lobe (if willing); and*

VI. $X < Y_1Y_21$ & $X \not\geq Y$ & $X \not\leq Y$: *can participate in exchange by donating a left lobe, or receive a direct right-lobe transplant (if willing).*

# Left or Right-Lobe Exchange: Feasibility



# Incentive Compatibility

- A mechanism is a systematic procedure that finds a matching for each willingness type profile reported.
- A mechanism is incentive compatible if it is a weakly dominant strategy for each pair to reveal its willingness truthfully.
- Since our mechanism will be based on a sequential algorithm, we will attain incentive compatibility by gradually transforming willing pairs as their left-lobe transplant prospects are fully exhausted.

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# Incentive Compatibility Based on Category

0. No direct transplant/exchange $\Longrightarrow$ Irrelevant: Remains w/o transplant

I. Direct left-lobe transplant only
$\Longrightarrow$ Irrelevant: Direct l-lobe transplant at the beginning

II. Direct right-lobe transplant only
$\Longrightarrow$ Transform for a direct r-lobe transplant at the beginning if willing

III. Exchange via left-lobe donation only
$\Longrightarrow$ Irrelevant: Role in the algorithm unaffected

IV. Exchange via right-lobe donation only
$\Longrightarrow$ Transform to donate a right lobe at the beginning if willing

V. Exchange via left-lobe or right-lobe donation
$\Longrightarrow$ **Gradually transform to donate a right lobe if willing, as left-lobe donation prospects are fully exhausted**

VI. Exchange via left-lobe donation, or direct right-lobe transplant
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If still unmatched, transform for a direct r-lobe transplant if willing

# Incentive Compatibility Based on Category

0. No direct transplant/exchange ⟹ Irrelevant: Remains w/o transplant

I. Direct left-lobe transplant only
⟹ Irrelevant: Direct l-lobe transplant at the beginning

II. Direct right-lobe transplant only
⟹ Transform for a direct r-lobe transplant at the beginning if willing

III. Exchange via left-lobe donation only
⟹ Irrelevant: Role in the algorithm unaffected

IV. Exchange via right-lobe donation only
⟹ Transform to donate a right lobe at the beginning if willing

V. Exchange via left-lobe or right-lobe donation
⟹ **Gradually transform to donate a right lobe if willing, as left-lobe donation prospects are fully exhausted**

VI. Exchange via left-lobe donation, or direct right-lobe transplant
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# Pareto Efficiency

- Insight from left-lobe-only exchange: Clear 0-waste, 1-waste, and then 2-waste exchanges, in this order, for efficiency.
- Build on the same insight, but integrating with our strategy for incentive compatibility.
- Pareto efficiency no longer implies transplant maximality. Indeed:

**Proposition**

*There is no incentive-compatible mechanism that maximizes*

1. *the number of transplants, or even*
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# Left or Right-Lobe Exchange: Algorithm

Fix a priority order over pairs.

**Step 0.** Direct transplant each Category I and Category II $w$ pair.

**Step 1.** Transform Category IV $w$ pairs.

Clear **0-waste** exchanges following the given priority order.

At least one of Category V types $010-100$ and $100-010$ is fully depleted. **Assume wlog type $100-010$ pairs are depleted.**

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No exchange remains for Category V type $011-100w$.
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No exchange remains for Category V type 011 – 100*w*.
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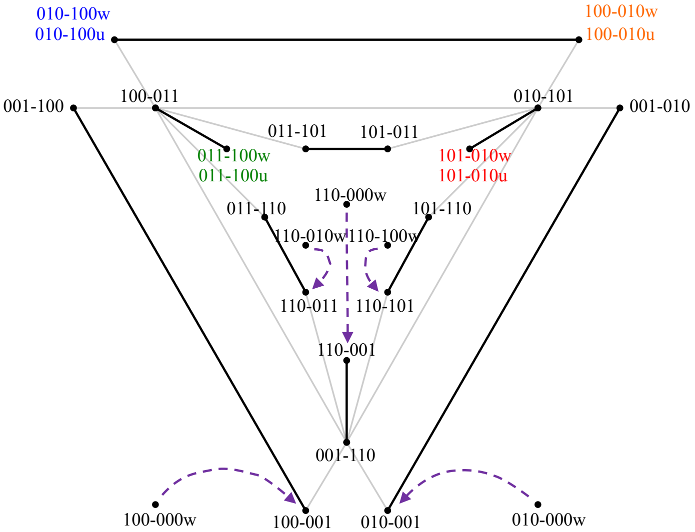
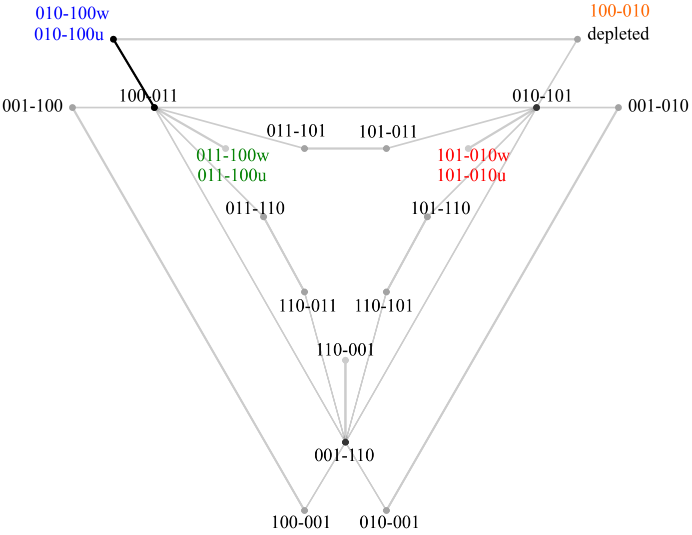
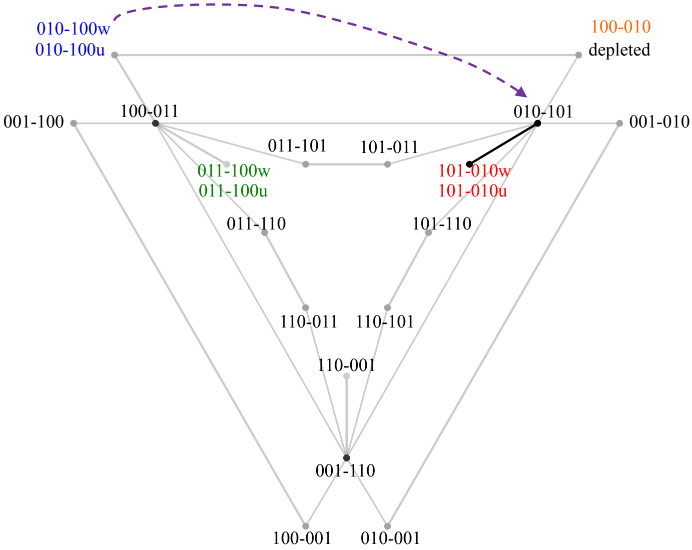
# Algorithm Step 1:



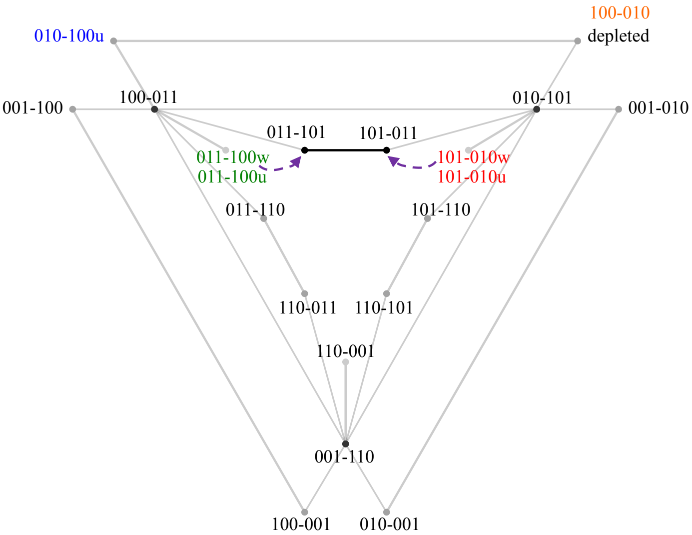
# Algorithm Step 2a:



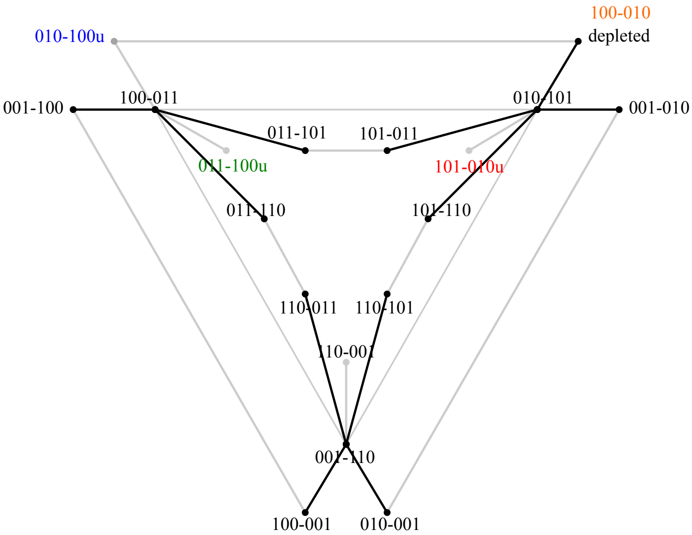
# Algorithm Step 2b:



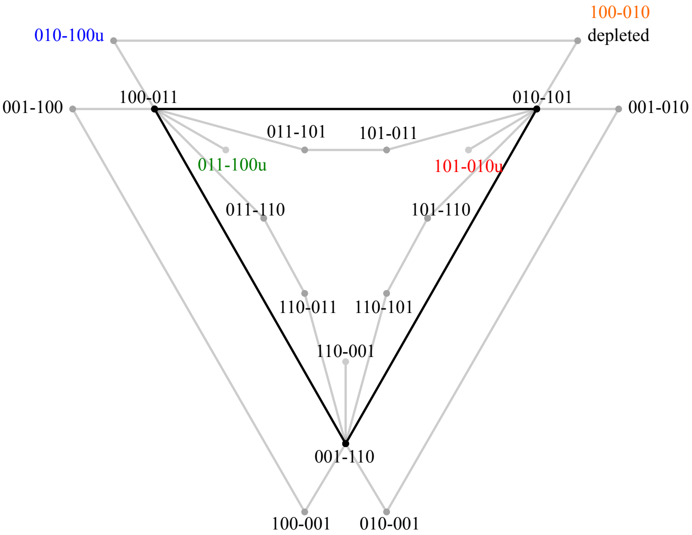
# Algorithm Step 2c:



# Algorithm Step 2d:



# Algorithm Step 3:



# Efficiency & Incentive Compatibility

**Theorem**

*The left or right-lobe sequential exchange mechanism is individually rational, Pareto-efficient, and incentive compatible.*

## Generalized Model: Multiple Individual Sizes

- $\mathcal{S} = \{0, 1, ..., S-1\}$: The set of possible patient/donor sizes
- Individual types: $X, Y \in \{0, 1\} \times \{0, 1\} \times \mathcal{S}$
- Pair types: $X - Y \in (\{0, 1\}^2 \times \mathcal{S})^2$
- Right-lobe donation function: A non-decreasing function $\rho : \mathcal{S} \to \mathcal{S}$ such that $\rho(s) > s$ for all $s \in \mathcal{S} \setminus \{S-1\}$

  A donor of size $s$ size can donate his right lobe to a blood-type compatible patient of any size $s' \leq \rho(s)$.
- Category V pairs: $X - Y$ such that $X \not\succeq Y$ & $X \not\preceq Y_1 Y_2 \rho(Y_3)$

# Difficulties with Generalization

1. Sequentially committing to an exchange may compromise efficiency, even for left-lobe-only exchange.
2. When right-lobe donation is possible, the transformation order of Category V willing pairs require further analysis.

# Transformation Order of Category V Pairs

- We will rely on a priority approach, based on matchability arguments.
- To maintain IC, it is plausible to transform a Category V pair after its left-lobe matchability options are exhausted.

But how does transformation of Category V pairs affect the matchability options of other Category V pairs?

**Definition**

Define the following precedence digraph on the set of Category V pair types, where for any Category V pair types $X - Y$ and $U - V$:

$$X - Y \longrightarrow U - V \quad \iff \quad X \leq V, \; U \not\leq Y \; \& \; U \leq \rho(Y).$$

If $X - Y \longrightarrow U - V$, we say that $X - Y$ precedes $U - V$.

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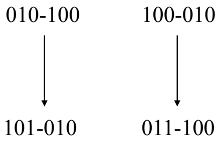
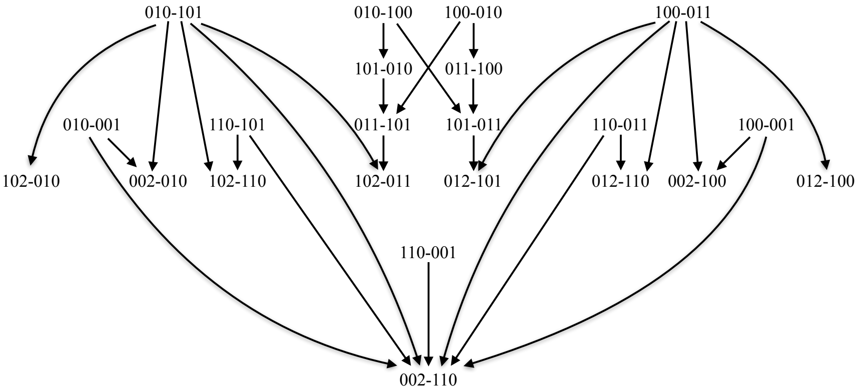
# Precedence Digaph: 2 Sizes



# Precedence Digaph: 3 Sizes



# Transformation Order of Category V Pairs

**Lemma (from graph theory)**

*Given an acyclic digraph, there exists a linear order of all nodes, known as a topological order, L, that is consistent with the digraph:*

$$x \to y \implies xLy$$

**Lemma**

*The precedence digraph on Category V pair types is acyclic.*

Thus, a topological order of Category V pair types, as well as a topological order of all Category V pairs exist.
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# Precedence-Order Induced Priority Mechanism

Fix a topological order over Category V pairs as $i_1, ..., i_K$ and a priority order over all pairs. Given a willingness profile $R$:

Step 0. Direct transplant Category I and Category III w pairs. *Transform* Category IV w pairs.

Step 1. Let $\mathcal{I}^0$ be the set of remaining pairs, $G^0$ be the current compatibility graph. Inductive:

*Step 1.k.* If next Category V Pair $i_k$ together with $\mathcal{I}^{k-1}$ are matchable in $G^{k-1}$, then $\mathcal{I}^k := \mathcal{I}^{k-1} \cup \{i_k\}$, $G^k := G^{k-1}$. Otherwise, set $\mathcal{I}^k := \mathcal{I}^{k-1}$, and if $i_k$ is willing, *transform* $i_k$ to obtain a new compatibility graph $G^k$ from $G^{k-1}$.

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*Step* 1.$k$. If next Category V Pair $i_k$ together with $\mathcal{I}^{k-1}$ are matchable in $G^{k-1}$, then $\mathcal{I}^k := \mathcal{I}^{k-1} \cup \{i_k\}$, $G^k := G^{k-1}$.
Otherwise, set $\mathcal{I}^k := \mathcal{I}^{k-1}$, and if $i_k$ is willing, *transform* $i_k$ to obtain a new compatibility graph $G^k$ from $G^{k-1}$.

Step 2. Let $j_1, ..., j_N$ be the remaining pairs in $\mathcal{I}^0 \setminus \mathcal{I}^K$ ordered wrt the priority order. Inductive:

*Step 2.n.* If next pair $j_n$ together with $\mathcal{I}^{K+(n-1)}$ are matchable in $G^K$, then let $\mathcal{I}^{K+n} := \mathcal{I}^{K+(n-1)} \cup \{j_n\}$. Otherwise, set $\mathcal{I}^{K+n} := \mathcal{I}^{K+(n-1)}$.

Step 3. Direct transplant willing Category VI pairs in $\mathcal{I}_0 \setminus \mathcal{I}^{K+N}$. Any matching in $G^K$ that matches all pairs in $\mathcal{I}^{K+N}$ in exchanges together with the fixed direct transplants is the outcome.

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# Generalized Model: Main Result

**Theorem**

*The precedence-order induced priority mechanism satisfies:*

- *individual rationality,*
- *Pareto efficiency, and*
- *incentive compatibility.*

# Generalized Model: Main Result

## Intuition of the Proof.

Individual rationality: By construction.

Pareto efficiency: Obtained by following

1. topological order for Category V pairs, and
2. priority order for remaining pairs and transformed Category V pairs.

Incentive compatibility: Acyclicity of the precedence digraph implies that *transformation* a willing Category V pair $i_k$ is independent of the willingness types of its lower-prioritized "graph neighbors."
Thus, they cannot affect how $i_k$ is matched by manipulating their own willingness types. □

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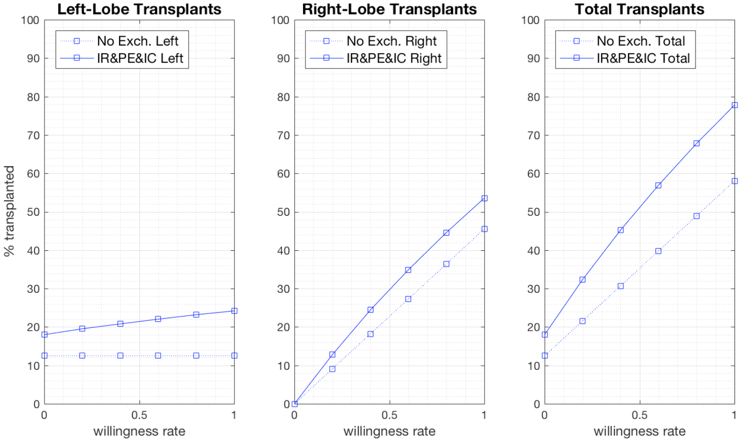
# Simulations

- Using South Korean population characteristics for $I = 100$
- % of left-lobe transplants higher under IR&PE&IC than no exchange
- IR&PE&IC generates 44%-34% more transplants than no exchange



# Conclusion

- We model living-donor liver exchange as a market design problem. Information/incentive problems are modeled and solved through a PE + IC mechanism.
- Size incompatibility increases the benefit from exchange, more gains plausible with respect to kidney exchange.
- Off-the-shelf-implementable mechanism in Middle East and East Asia: Liver transplants are more complex, two-way may be the way to start the exchange.
- Implications for matching theory in general: A new class of bilateral exchange mechanisms for *n*-dimensional vector partial-order induced weak preferences:
  - Other examples: vacation house exchanges, time/favor exchanges
  - Two-size model with three dimensions is of independent interest: Induces a fully-symmetric model where greedy mechanism design is possible.