Housing Markets & House Allocation

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House Allocation Problems: A Collective Ownership Economy

A **house allocation problem** (Hylland & Zeckhauser, *JPE* 1979) is a triple $\langle I, H, \succ \rangle$.

I: set of agents

H: set of houses

$\succ$: list of preferences over houses

For simplicity assume:

1. $|H| = |I|$, and
2. the preferences are strict.

The Outcome: A Matching

- A **(house) matching** $\mu : I \to H$ is a one-to-one and onto function from I to H.

  With everyday language it is an assignment of houses to agents such that

  1. every agent is assigned one house, and
  2. no house is assigned to more than one agent.

- A matching μ **Pareto dominates** another matching ν if

  1. $\mu(i) \succeq_i \nu(i)$ for all $i \in I$ and
  2. $\mu(i) \succ_i \nu(i)$ for some $i \in I$.

- A matching is **Pareto efficient** if it is not Pareto dominated by any other matching.

Housing Markets: A Basic Exchange Economy

A **housing market** (Shapley & Scarf, *JPE* 1974) is a 4-tuple $\langle I, H, \succ, \mu \rangle$.

I: set of agents

H: set of houses with $|H| = |I|$

$\succ$: list of strict preferences over houses

μ: initial endowment matching

Let $h_i = \mu(i)$ denote the initial endowment of agent $i \in I$.

- A matching η is **individually rational** if

$$\eta(i) \succeq_i h_i \qquad \text{for all } i \in N.$$

- A matching η is in the **core** of the housing market $(I, H, \succ, \mu)$ if there is no coalition $T \subseteq I$ and matching ν such that

  1. $\nu(i) \in \{h_i\}_{i \in T} \qquad$ for all $i \in T$,
  2. $\nu(i) \succeq_i \eta(i) \qquad$ for all $i \in T$,
  3. $\nu(i) \succ_i \eta(i) \qquad$ for some $i \in T$.

Gale's Top Trading Cycles Algorithm

(Described in Shapley & Scarf, attributed to David Gale)

Step 1: Each agent "points to" the owner of his favorite house. Since there are finite number of agents, there is at least one **cycle**.

Each agent in a cycle is assigned the house of the agent he points to and removed from the market with his assignment.

If there is at least one remaining agent, proceed with the next step.

Step t: Each remaining agent points to the owner of his favorite house among the remaining houses.

Every agent in a cycle is assigned the house of the agent he points to and removed from the market with his assignment.

If there is at least one remaining agent, proceed with the next step.

Important Properties of the Core

- **Theorem** (Roth & Postlewaite, *JME* 1977): The outcome of Gale's TCC algorithm is the unique matching in the core of each housing market. Moreover, this matching is the unique competitive allocation.

Sketch of Key Elements of the Proof:

Uniqueness: Agents who leave in Step 1 has to receive their top choices for otherwise they will block. Subject to that, agents who leave in Step 2 has to receive their top choices among remaining choices for otherwise they will block. Proceeding in a similar way, each agent should receive her outcome under Gale's TTC algorithm.

Core is a Competitive Allocation: Suppose the algoritm terminates in T steps. Here is a competitive price:

- The price of each house that leaves the algorithm in Step 1 is T,
- the price of each house that leaves the algorithm in Step 2 is $T - 1$,
- $\vdots$ $\vdots$
- the price of each house that leaves the algorithm in Step T is 1.

- A **direct matching mechanism** is a systematic procedure to select a matching for each problem.

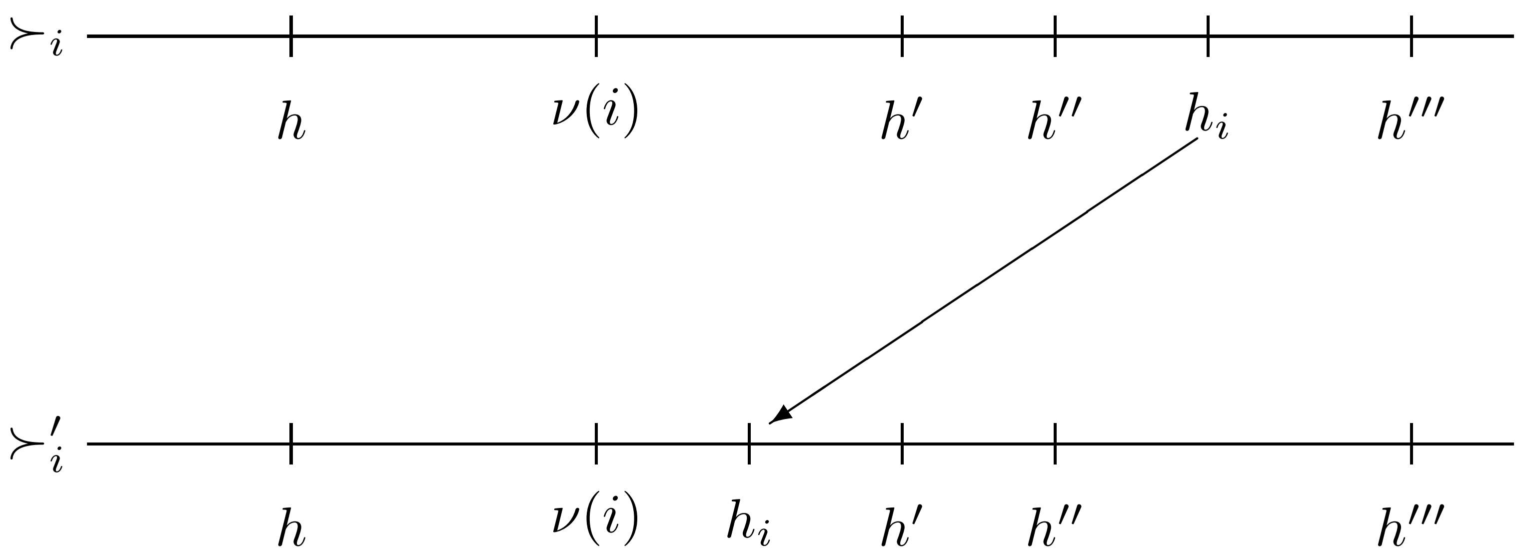
- A direct mechanism is **strategy-proof** if truth-telling is a dominant strategy in the resulting preference revelation game.

- **Theorem** (Roth, *Economics Letters* 1982): Core (as a direct mechanism) is *strategy-proof.*

  Sketch of the Proof: Consider Gale's TTC algorithm. Suppose an agent leaves the algorithm with her assignment in Step t. She cannot stop the formation of cycles that form before Step t by misrepresenting her preferences. (These cycles only depend on preferences of agents who are in those cycles.) So she cannot receive a better assignment through a preference manipulation.

- **Theorem** (Ma, *IJGT* 1994): Core is the only mechanism that is *Pareto efficient*, *individually rational*, and *strategy-proof*.

  Sketch of the Proof: Let ν be the matching in the core for housing market $\langle I, H, \succ, \mu \rangle$. Construct the following preference relation $\succ'_i$ for each agent i:

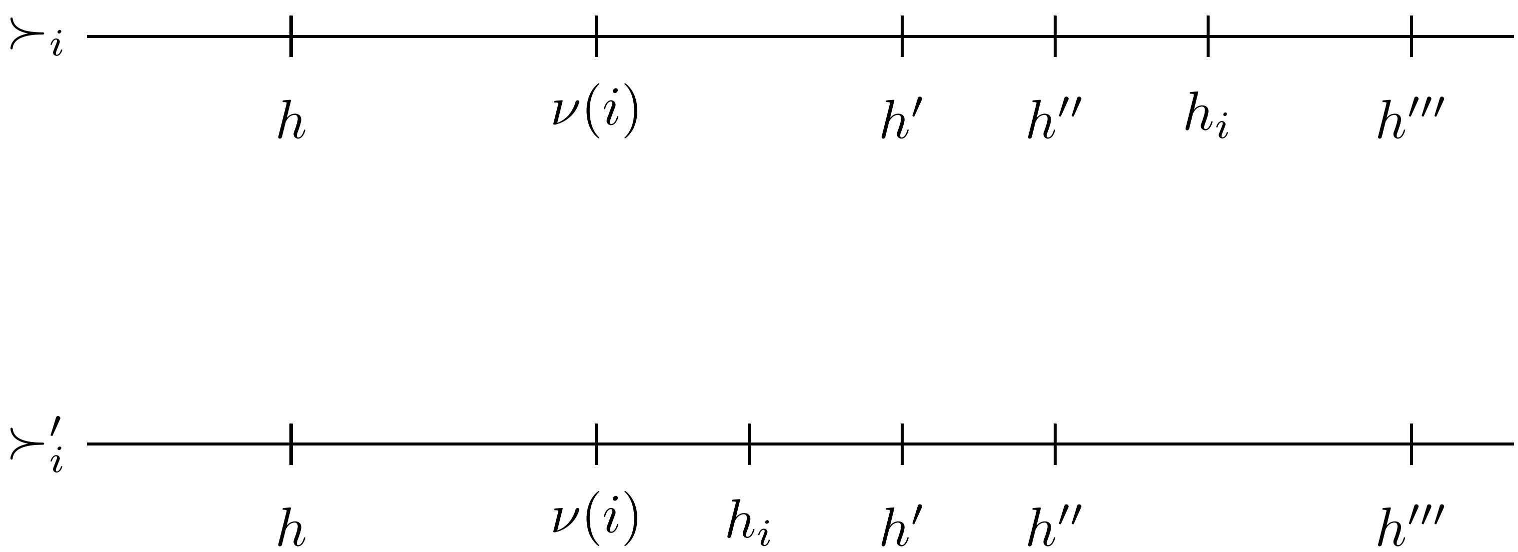


Let ϕ be a *PE, IR,* and *S-P* mechanism.

Claim 1: $\phi(\succ') = \nu$ because ν is the only PE and IR matching under $\succ'$.

Claim 2: $\phi(\succ) = \nu$.

Replace with true preferences one agent at a time:



Mechanisms for House Allocation

An **ordering** $f : \{1, \ldots, n\} \to I$ is a one-to-one and onto function.

Simple serial dictatorship induced by f: Agent who is ordered first (by the ordering f) gets her top choice; agent ordered second gets his top choice among those remaining; and so on.

Core from assigned endowments μ: For any house allocation problem $\langle I, H, \succ \rangle$, select the core of the housing market $\langle I, H, \succ, \mu \rangle$.

Lottery Mechanisms

A **lottery** is a probability distribution over matchings.

A **direct lottery mechanism** is a systematic procedure to select a lottery for each problem.

Examples:

- **Random serial dictatorship (RSD)**: Randomly select an ordering with uniform distribution and use the induced simple serial dictatorship.
- **Core from random endowments**: Randomly select a matching (to be interpreted as initial endowment) and select the core of the induced housing market.

Efficient House Allocation

Theorem (Abdulkadiroğlu & Sönmez, *Econometrica* 1998): For any ordering f, and any matching μ, simple serial dictatorship induced by f and core from assigned endowments μ both yield *Pareto efficient* matchings. Moreover, for any *Pareto efficient* matching η, there is a *simple serial dictatorship* and a *core from assigned endowments* that yields it.

Theorem (Abdulkadiroğlu & Sönmez, *Econometrica* 1998): The *random serial dictatorship* is equivalent to the the *core from random endowments.* That is, for each house allocation problem they choose the same lottery.

A Slight Variation: One Existing Tenant

- A number of houses should be allocated to a group of agents through RSD.
- One of the houses is already occupied and its tenant is given two options:
  1. Keep the house, or
  2. Give it up and enter the lottery.
- Since there are no guarantees to get a better house, the existing tenant may choose the first option which in turn may result in loss of potential gains from trade.

Example 1: There are three agents i_1, i_2, i_3 and three house h_1, h_2, h_3. Agent i_1 is a current tenant and he occupies house h_1. Agents i_2, i_3 are new applicants and house h_2, h_3 are vacant houses.

- Utilities are:

	h_1	h_2	h_3
i_1	3	4	1
i_2	4	3	1
i_3	3	4	1

- Agent i_1 has two options:
  1. he can keep house h_1 or
  2. he can give it up and enter the lottery.
- His utility from keeping house h_1 is 3.

- The following table summarizes the possible outcomes, in case he enters the lottery:

ordering	i_1	i_2	i_3
$i_1 - i_2 - i_3$	h_2	h_1	h_3
$i_1 - i_3 - i_2$	h_2	h_3	h_1
$i_2 - i_1 - i_3$	h_2	h_1	h_3
$i_2 - i_3 - i_1$	h_3	h_1	h_2
$i_3 - i_1 - i_2$	h_1	h_3	h_2
$i_3 - i_2 - i_1$	h_3	h_1	h_2

- Expected utility from entering the lottery:

$$\frac{1}{6}u(h_1) + \frac{3}{6}u(h_2) + \frac{2}{6}u(h_3) = \frac{3}{6} + \frac{12}{6} + \frac{2}{6} = \frac{17}{6}.$$

- Optimal strategy: Keep house h_1.

When i_1 Keeps h_1:

- Since both i_2, i_3 prefer h_2 to h_3, the eventual outcome is either

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ h_1 & h_2 & h_3 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} i_1 & i_2 & i_3 \\ h_1 & h_3 & h_2 \end{pmatrix}$$

  both with 1/2 probability.

- **Inefficiency**: The first one is Pareto dominated by

$$\begin{pmatrix} i_1 & i_2 & i_3 \\ h_2 & h_1 & h_3 \end{pmatrix}$$

Avoiding Inefficiency with One Existing Tenant

The cause for the inefficiency is the lack of the mechanism to guarantee the existing tenant a house that is at least as good as the one he already holds. One natural modification that will fix this "deficiency" is the following:

1. Order the agents with a lottery.
2. Assign the first agent his top choice, the second agent his top choice among the remaining houses, and so on, *until someone demands the house the existing tenant holds.*
3. (a) If the existing tenant is already assigned a house, then do not disturb the procedure.
   (b) If the existing tenant is not assigned a house, then modify the remainder of the ordering by inserting him to the top, and proceed with the procedure.

A General Model: House Allocation with Existing Tenants

- A set of houses should be allocated to a set of agents by a centralized clearing house.
- Some of the agents are existing tenants each of whom already occupies a house and the rest of the agents are newcomers.
- In addition to occupied houses, there are vacant houses.
- Existing tenants are not only entitled to keep their current houses but also apply for other houses.

Formal Model

A **house allocation problem with existing tenants** (Abdulkadiroğlu & Sönmez *JET* 1999) is a five-tuple $\langle I_E, I_N, H_O, H_V, \succ \rangle$ where

1. I_E is a finite set of existing tenants,
2. I_N is a finite set of newcomers,
3. $H_O = \{h_i\}_{i \in I_E}$ is a finite set of occupied houses,
4. H_V is a finite set of vacant houses where $h_0 \in H_V$ denotes the null house, and
5. $\succ = (\succ_i)_{i \in I_E \cup I_N}$ is a list of strict preference relations.

- Assume that the "null house" h_0 (i.e receiving nothing) is the last choice for each agent.

- A **matching** μ is an assignment of houses to agents such that
  1. every agent is assigned one house, and
  2. only the null house h_0 can be assigned to more than one agent.
- A matching is **Pareto efficient** if there is no other matching which makes all agents weakly better off and at least one agent strictly better off.
- A matching is **individually rational** if no existing tenant strictly prefers his endowment to his assignment.

A Popular Real-Life Mechanism: RSD with squatting rights

- Each existing tenant decides whether she will enter the housing lottery or keep her current house. Those who prefer keeping their houses are assigned their houses. All other houses become available for allocation.
- An ordering of agents in the lottery is randomly chosen from a given distribution of orderings. This distribution may be uniform or it may favor some groups.
- Once the agents are ordered, available houses are allocated using the induced *simple serial dictatorship*: The first agent receives her top choice, the next agent receives her top choice among the remaining houses and so on so forth.

Major deficiency: Neither *individually rational* nor *Pareto efficient*.

Solution: Top Trading Cycles Mechanism

Fix an ordering f of agents. Interpret this as a **priority ordering**.

Step 1: Define the set of **available houses** for this step to be the set of vacant houses.

* Each agent points to his favorite house,
* each occupied house points to its occupant,
* each available house points to the agent with highest priority.

- There is at least one cycle. Every agent in a cycle is assigned the house that he points to and removed from the market with his assignment.
- If there is at least one remaing agent and one remaining house then we go to the next step.

TTC: Adjustment of Available Houses

- Whenever there is an available house in a cycle, the agent with the highest priority, i.e. agent $f(1)$, is also in the same cycle.
- If this agent is an existing tenant, then his house $h_{f(1)}$ cannot be in any cycle and it becomes available for the next step.
- All available houses that are not removed remain available.

Step t: The set of **available houses** for Step t is defined at the end of Step (t-1).

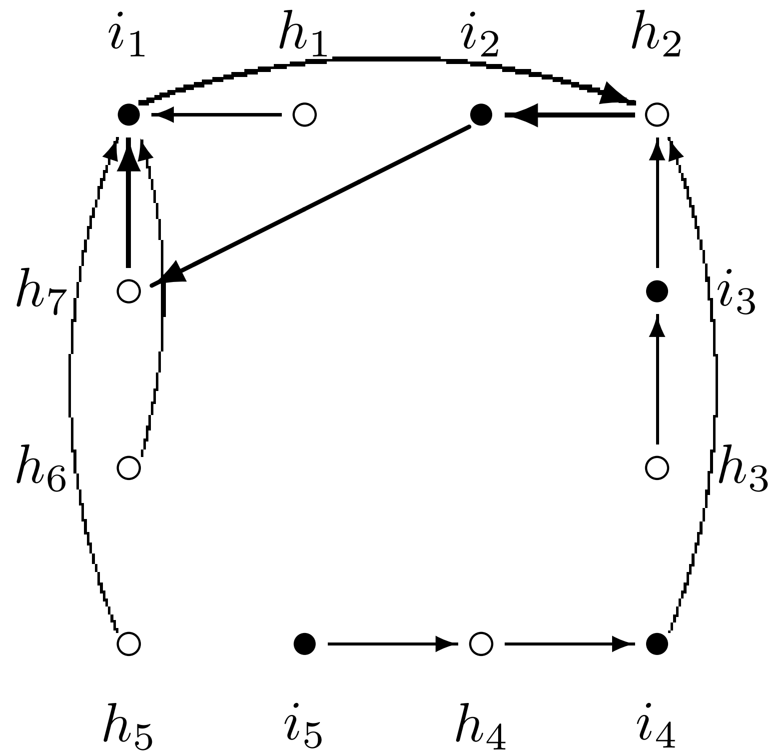
  * Each remaining agent points to his favorite house among the remaining houses,
  * each remaining occupied house points to its occupant,
  * each available house points to the agent with highest priority among the remaing agents.

- Every agent in a cycle is assigned the house that he points to and removed from the market with his assignment.
- If the most senior (remaining) agent's house is vacated, then it is added to the set of available houses for the next step. All available houses that are not removed remain available.
- If there is at least one remaing agent and one remaining house then we go to the next step.

Example 2: Let $I_E = \{i_1, i_2, i_3, i_4\}$, $I_N = \{i_5\}$, $H_O = \{h_1, h_2, h_3, h_4\}$, and $H_V = \{h_5, h_6, h_7\}$. Let the ordering f order the agents as $i_1 - i_2 - i_3 - i_4 - i_5$ and the preferences (from best to worst) be as follows:

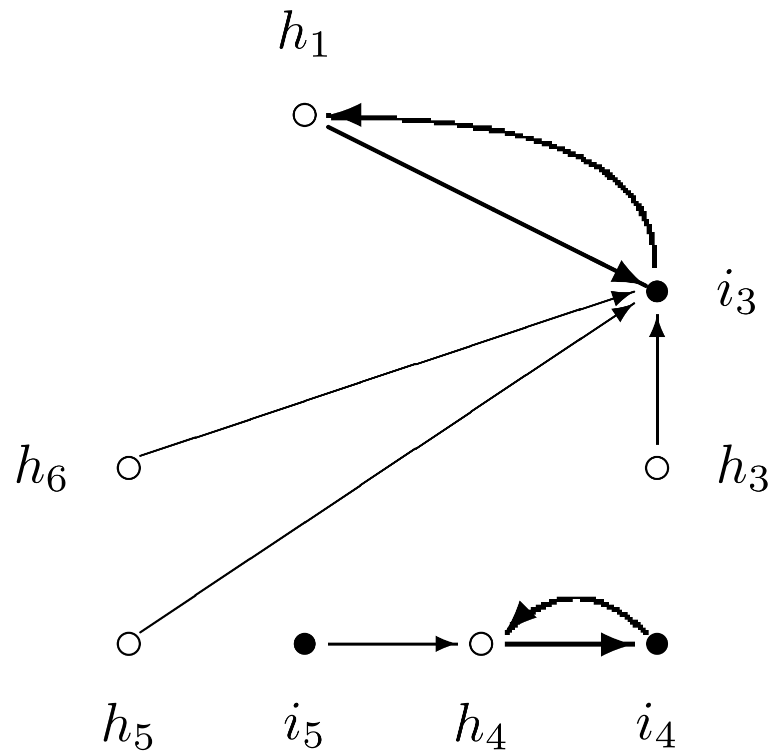
$\succ_{i_1}$	$\succ_{i_2}$	$\succ_{i_3}$	$\succ_{i_4}$	$\succ_{i_5}$
h_2	h_7	h_2	h_2	h_4
h_6	h_1	h_1	h_4	h_3
h_5	h_6	h_4	h_3	h_7
h_1	h_5	h_7	h_6	h_1
h_4	h_4	h_3	h_1	h_2
h_3	h_3	h_6	h_7	h_5
h_7	h_2	h_5	h_5	h_6
h_0	h_0	h_0	h_0	h_0

Step 1:



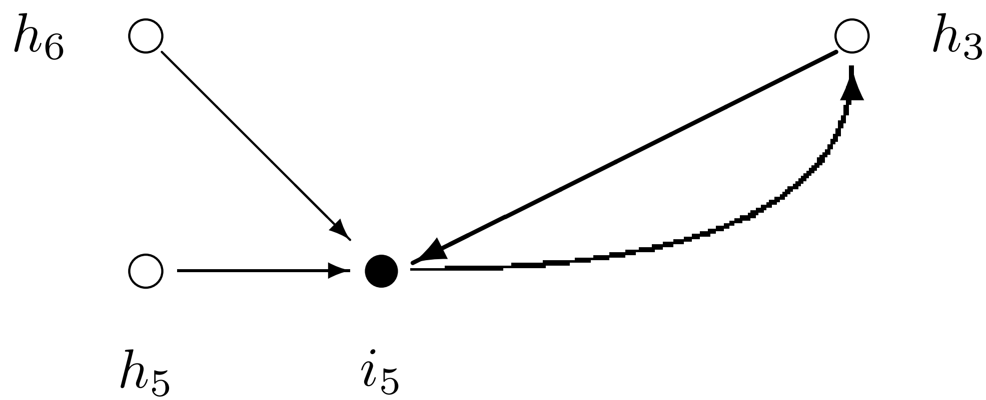
- The set of available houses in Step 1 is $H_V = \{h_5, h_6, h_7\}$.
- The only cycle that is formed in this step is (i_1, h_2, i_2, h_7). Therefore i_1 is assigned h_2 and i_2 is assigned h_7.
- Since i_1 leaves the market, his house h_1 becomes available for the next step. Therefore the set of available houses for Step 2 is $\{h_1, h_5, h_6\}$.

Step 2:



- There are two cycles (i_3, h_1) and (i_4, h_4) in Step 2.
- Therefore i_3 is assigned h_1 and i_4 is assigned h_4.
- Since i_3 leaves the market his house h_3 becomes available for the next step. Therefore the set of available houses for Step 3 is $\{h_3, h_5, h_6\}$.

Step 3:



- There is one cycle (i_5, h_3) in Step 3.
- Therefore i_5 is assigned h_3.
- There are no remaining agents so the algorithm terminates and the matching it induces is:

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 \\ h_2 & h_7 & h_1 & h_4 & h_3 \end{pmatrix}$$

Efficiency, Individual Rationality, and Strategy-Proofness

- TTC reduces to Gale's TTC for housing markets and RSD for house allocation problems.

- Moreover:

  Theorem (Abdulkadiroğlu & Sönmez *JET* 1999): For any ordering f, the induced *top trading cycles mechanism* is

  * individually rational,
  * Pareto efficient, and
  * strategy-proof.

Simpler Efficient, Individual Rational, and Strategy-Proof Mechanism

- Consider the case where: $|H_V| = |I_N|$ (so that there are same number of agents and houses).
- Simpler PE, IR and S-P mechanism:
  1. Construct an initial allocation by
     (a) assigning each existing tenant her own house and
     (b) randomly assigning the vacant houses to newcomers with uniform distribution, and
  2. choose the core of the induced housing market to determine the final outcome.
- **Theorem** (Sönmez & Ünver *GEB* 2005): The above mechanism is equivalent to an extreme case of TTC where newcomers are randomly ordered first and existing tenants are randomly ordered next.

A More Intuitive Algorithm

We can find the outcome of the TTC mechanism using the following **you ask my house-I get your turn (YRMH-IGYT) algorithm**:

1. For any given ordering f, assign the first agent his top choice, the second agent his top choice among the remaining houses, and so on, until someone demands the house of an existing tenant.

2. If at that point the existing tenant whose house is demanded is already assigned a house, then do not disturb the procedure.

   Otherwise modify the remainder of the ordering by inserting him to the top and proceed with the procedure.

3. Similarly, insert any existing tenant who is not already served at the top of the line once his house is demanded.

4. If at any point a loop forms, it is formed by exclusively existing tenants and each of them demands the house of the tenant next in the loop. (A **loop** is an ordered list of agents $(i_1, i_2, \ldots, i_k)$ where agent i_1 demands the house of agent i_2, agent i_2 demands the house of agent i_3,…, agent i_k demands the house of agent i_1.) In such cases remove all agents in the loop by assigning them the houses they demand and proceed with the procedure.

- **Theorem** (Abdulkadiroğlu & Sönmez *JET* 1999): For a given ordering f, YRMH-IGYT algorithm yields the same outcome as the TTC algorithm.