Reserve Design: Unintended Consequences and The Demise of Boston's Walk Zones*

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Abstract

Admissions policies often use reserves to grant certain applicants higher priority for some (but not all) available seats. Boston's school choice system, for example, reserved half of each school's seats for local neighborhood applicants while leaving the other half for open competition. This paper shows that in the presence of reserves, the effect of the *precedence order* (i.e., the order in which different types of seats are filled) on distributional objectives is comparable to the effect of adjusting reserve sizes. Either lowering the precedence order positions of reserve seats at a school or increasing the number of reserve seats weakly increases reserve-group assignment at that school. Using data from Boston, we show that reserve and precedence adjustments have similar quantitative effects. Our results illustrate that policies about precedence, heretofore under-explored, are inseparable from other aspects of admissions policy. Moreover, our findings explain the puzzling empirical fact that despite careful attention to the importance of neighborhood priority, Boston's implementation of its 50-50 reserve–open seat split was nearly identical to the outcome of a counterfactual system without any reserves. Transparency about these issues—in particular, how precedence unintentionally undermined the intended admissions policy—led to the elimination of Boston's walk zones.

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1 Introduction

Admissions policies in school systems are often shaped by historical circumstances and modern-day compromises between competing interest groups. At many publicly-funded Indian engineering colleges, for example, seats are reserved for applicants from disadvantaged caste and gender groups (see Bagde, Epple, and Taylor (2016)). In the Indian system, an applicant from a disadvantaged group who qualifies for a school without invoking caste/gender priority is assigned one of the school's regular seats instead of a reserve seat; the reserve seats are held for students who otherwise would not be able to gain admission. The public school administration in Boston also devised a reserve scheme, but based it on neighborhood boundaries rather than student types. The Boston policy came after 1970s-era court ordered desegregation divided the city into geographically segregated communities. At each school in Boston, half of the seats at each school were made open to all applicants, while the other half prioritized applicants from local neighborhoods. Unlike the Indian system, the Boston system filled reserve seats ahead of open seats.

Indian engineering admissions are decentralized in some states, while Boston's school choice program is centralized. Under both systems, however, there are two types of seats at each school—reserve seats and open seats—and it is common for applicants to qualify for both seat types. When a student can be admitted to a school via multiple routes, an admissions policy must specify the relative *precedence* of different admissions tracks; in the cases of Indian engineering colleges and Boston public schools, this means that policy must account for the order in which reserve and open seats are processed. In this paper, we formally show that precedence plays a central role that is qualitatively and quantitatively similar to the impact of reserve sizes in achieving distributional objectives. We then relate these results to a recent policy discussion in Boston, showing how an oversight leading to the wrong precedence policy completely undermined the city's stated objectives in a subtle way.

Boston's 50-50 reserve–open seat split emerged from a city-wide discussion after racial and ethnic criteria for school placement ended in 1999. Many stakeholders advocated abandoning school choice and returning to neighborhood schooling, but the public school committee chose instead to maintain school choice while making neighborhood, "walk-zone" priority apply to 50% of each school's seats (Appendix C excerpts the official policy). In popular accounts, the 50-50 seat split was described as "striking an uneasy compromise between neighborhood school advocates and those who want choice," while the Superintendent hoped the "plan would satisfy both factions, those who want to send children to schools close by and those who want choice" (Daley 1999).

The fragile compromise between the pro-neighborhood schooling faction and the pro-school choice faction has resurfaced in numerous debates about Boston's school admissions policies.[1] Boston Mayor Thomas Menino's 2012 State of the City Address forcefully argued in favor of as-

[1] A December 2003 community engagement process in Boston considered six different proposals for alternative neighborhood zone definitions. However, the only recommendation adopted by the school committee was to switch the assignment algorithm (Landsmark and Dajer 2004, Abdulkadiroğlu, Pathak, Roth, and Sönmez 2005). In 2009, BPS renewed the discussion with a proposal for a five-zone plan, that eventually was not approved (Vaznis 2009).

signing students to schools closer to home (Menino 2012).[2] Proposals from Boston Public Schools (BPS) and other community members became the center of a year-long, city-wide school choice discussion that featured over seventy public meetings and input from more than 3,000 parents.[3] Boston's decision to revisit its reserve policy was partly motivated by a persistent empirical puzzle: while 50% of seats at each school were reserved for students living in the neighborhood/walk-zone, the fraction of neighborhood students assigned to popular schools consistently hovered around 50%. With half the seats reserved for neighborhood students and the other half open to everyone, one would expect more than 50% neighborhood assignment, as Boston's official policy suggests (see Appendix C).

In this paper, we show that Boston's assignment puzzle was an unintended consequence of the chosen implementation of the walk-zone reserve: because the precedence order filled reserve seats before open seats, the 50-50 compromise was completely subverted, resulting in an allocation almost indistinguishable from a counterfactual setting without any reserve seats at all.

Our first formal result shows that reserves and precedence are policy tools with similar qualitative effects for any given school. For any precedence order, replacing an open slot with a reserve slot weakly increases the assignment of reserve-eligible applicants. Similarly, for any given reserve size, swapping the precedence-order position of a reserve slot with that of lower-precedence open slot weakly increases the assignment of reserve-eligible applicants. Next, we investigate how our within-school results extend to centralized assignment systems that use the deferred acceptance algorithm. We find that for a given school, increasing the number of reserve slots (relative to open slots) or raising the precedence of open slots (relative to reserve slots) increases admission of reserve-eligible applicants under deferred acceptance. This result is, to our knowledge, the first-ever comparative static result for multi-agent priority improvements in matching models. Because of interactions across schools in the deferred acceptance algorithm, the comparative statics we find do not necessarily extend to aggregate increases in assignment of reserve-eligible applicants across all schools. However, even though pathological cross-school interactions are possible, they do not appear to be relevant in practice: Our comparative statics extend to the whole market in a two-school model, and we can also bound the worst case when reserves privilege the same group throughout the school system. Moreover, our theoretical analysis closely matches the empirical patterns observed in Boston—we show that Boston's implementation of the 50-50 reserve–open compromise was in practice closer to a 10-90 system once implemented.

This paper contributes to a broader agenda, examined in a number of recent papers, that introduces diversity concerns into the literature on school choice mechanism design (see, e.g., Erdil

[2]Constituents had long believed that students were traveling too far to attend schools, and sought to alter the plan to assign students to schools closer to home (Landsmark 2009).

[3]For more on this debate, see the materials available at http://bostonschoolchoice.org and accounts by Goldstein (2012), Handy (2012), and Seelye (2012). In Fall 2012, BPS proposed five different plans that all restricted participant choice by reducing the number of schools that students could rank; the idea behind these plans was to reduce the fraction of non-neighborhood applicants competing for seats at each school. (The initial plans suggested dividing the city into 6, 9, 11, or 23 zones, or doing away with school choice entirely and reverting to assignment based purely on neighborhood.)

and Kumano (2012), Kojima (2012), Budish, Che, Kojima, and Milgrom (2013), Hafalir, Yenmez, and Yildirim (2013), Kominers and Sönmez (2013, 2016), Echenique and Yenmez (2015)). When an applicant ranks a school with seats that employ different admissions criteria, it is as if she is indifferent between those school's seats. Therefore, our work parallels investigations of indifferences in school choice problems (see, e.g., Erdil and Ergin (2008), Abdulkadiroğlu, Pathak, and Roth (2009), Pathak and Sethuraman (2011)). Yet, results on school-side indifferences do not extend to indifferences in student preferences. Finally, our goal here is to establish comparative static results based on Boston's policy developments. In subsequent work, Dur, Pathak, and Sönmez (2016) characterized optimal admissions policies motivated by Chicago's place-based affirmative action system.

Our paper proceeds as follows. Section 2 describes the puzzle Boston faced in more detail. Section 3 formally studies admissions policies in which applicants can be admitted via multiple routes. Section 4 examines how schools' admissions policies interact with a centralized admissions system based on deferred acceptance. Section 5 reports on data from Boston, and Section 6 concludes. All proofs are presented in the Appendix.

2 Motivation

2.1 Boston's "50-50 Puzzle"

Despite widespread perception and policy intent that, since 1999, the BPS school choice system had prioritized walk-zone applicants, those applicants appear to have had little advantage in practice. Even though 50% of seats at each school were reserved for walk-zone students, the assignment outcomes were close to those that would have arisen under a system without any walk-zone reserve. To see this, we compute the fraction of students assigned to walk-zone schools in Boston for the extreme case with no walk-zone priority—the **0% Walk** system.[4] Table 1 shows that despite the 50% walk-zone reserve, assignment outcomes under BPS's system are nearly identical to those under 0% Walk; they differ for only 3% of Grade K1 students.

One might suspect that similarity between the BPS outcome and 0% Walk is driven by strong preferences among applicants for neighborhood schools, as such preferences would bring the two policies' outcomes close together. However, this is not the case. We compare the BPS outcome to a **100% Walk** counterfactual in which all seats give priority to walk-zone applicants. Under 100% Walk, 19% of Grade K1 students obtain a different assignment than they receive in the BPS outcome.[5] Thus, the remarkable proximity of the BPS outcome and the 0% Walk ideal of school

[4]To compute counterfactual assignments, we use internal preference data from BPS, and the same lottery numbers BPS used to break ties in its assignment system. It is worth noting that strategy-proofness (i.e., truthfulness) of the assignment mechanism used in Boston justifies re-computing the assignment without modeling how applicants might submit preferences under counterfactual mechanisms (see Abdulkadiroğlu, Pathak, Roth, and Sönmez (2006), Pathak and Sönmez (2008), Abdulkadiroğlu, Pathak, and Roth (2009) and Agarwal, Abdulkadiroğlu, and Pathak (2015)).

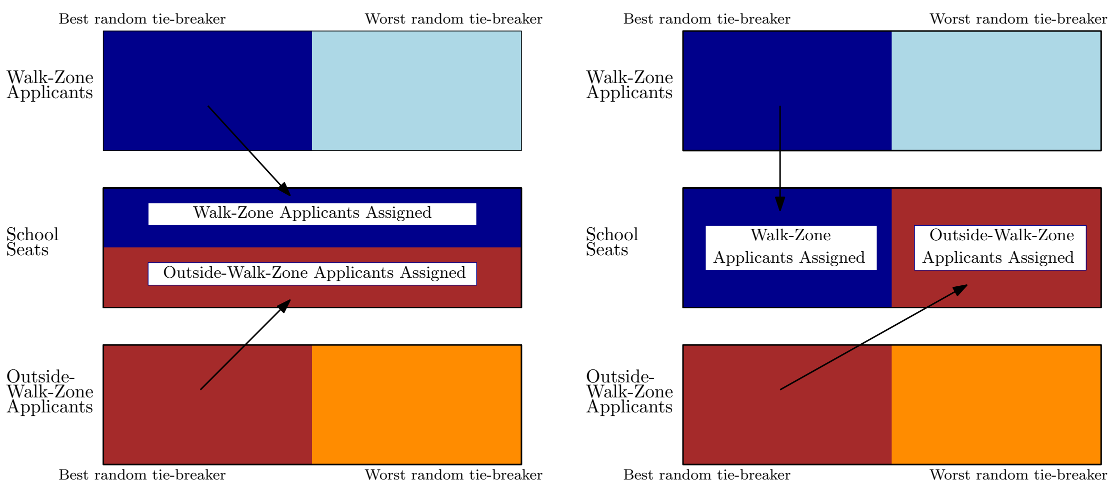
[5]We have repeated our calculations for 500 different random lottery draws. Under 0% Walk, the average differences are 3%, 4%, and 1% for Grades K1, K2, and Grade 6, respectively. Under 100% Walk, the corresponding average

choice proponents neither suggests nor reflects negligible stakes in school choice.[6] Rather, it presents a puzzle: Why does Boston's assignment mechanism result so closely resemble that of a system without any neighborhood priority, even though half of each school's seats prioritize neighborhood students? Or, more qualitatively, why did Boston's 50% reserve have so little impact in practice? Why did the policy not result in, say, an outcome half-way between 0% Walk and 100% Walk? To obtain intuition, we turn to a simple, single-school example that illustrates Boston's 50-50 seat split as implemented.

2.2 A One-School Example

Consider a single school with 100 seats. Suppose there are 100 applicants with walk-zone priority and 100 applicants without walk-zone priority. A lottery used for tie-breaking is such that, of the 100 applicants with highest lottery numbers, 50 are from the walk zone and 50 are not. Figure 1 illustrates the situation, with both walk-zone applicants and non walk-zone applicants ordered by the random tie-breaker.

In Panel (a) of Figure 1, there is no walk-zone priority at the school, so students are admitted solely based on the random tie-breaker. Given the tie-breaker, the school admits an equal number of students from each group. That is, the school admits 50 students from the walk zone and 50 students from outside the walk zone.



(a) Assignment without walk-zone priority.

(b) Assignment under Boston's implementation of the 50-50 seat split.

Figure 1: A single-school illustration comparing assignment without walk-zone priority to Boston's implementation of a 50-50 walk-zone–open seat split.

differences are 20%, 18%, and 10%.

[6]The patterns we observe are similar for grades above K1, with the smaller differences between 0% Walk and 100% Walk for higher grades driven by a larger share of continuing students who obtain guaranteed priority for higher grades.

In Panel (b) of Figure 1, half the seats grant walk-zone priority and the other half are open. Under Boston's school choice system, students from both groups first apply to the walk-zone half. For the walk-zone half, students who have walk-zone priority are admitted ahead of students who do not, and the admitted walk-zone students are those with the most favorable random tie-breakers. Therefore, 50 students from the walk zone with the most favorable tie-breakers take up all of the seats in the walk-zone half. Next, the remaining applicants from the walk zone—who have less favorable random tie-breakers—apply to the open half of the school together with all applicants from outside the walk zone. For the open seats, students are admitted based only on the random tie-breaker. But at this point, the remaining walk-zone applicants are disadvantaged because they have systematically less favorable tie-breakers; consequently, only non-walk-zone applicants are assigned to the 50 seats in the open half. The final allocation results in half of the school's seats being assigned to walk-zone applicants, with the remaining half assigned to applicants from outside the walk zone.

The preceding logic, illustrated in Figure 1 shows how the 50-50 compromise can have the same outcome as a situation without any walk-zone priority. However, our example is stylized in several ways: There are an equal number of applicants with walk-zone priority and without it,[7] and the tie-breaker has an equal number of students from each group among the top 100.[8] Nevertheless, we capture the main intuition for the phenomenon documented in Table 1.

Our example shows that the precedence order under which seats are processed significantly affects the outcome. Had all the applicants first applied to the open half, 75 walk-zone applicants and 25 non-walk-zone applicants would have been admitted—even holding fixed the 50-50 seat split. At the time of Menino's 2012 speech, precedence order's dramatic role in disadvantaging walk-zone students came as a surprise to many—including us—and motivated the formal analysis we now describe.

3 Admissions Policies with Reserves

To formalize the intuition presented in the preceding section, we develop a model of school admissions policies in which some seats at each school may be reserved for members of distinguished groups (e.g., disadvantaged castes or walk-zone students). We prove comparative statics illustrating that both (1) increasing the number of reserve seats and (2) raising the precedence order positions of open seats will (weakly) increase the number of reserve-eligible students who are accepted.

[7] It is easy to see that the same arguments work whenever there are more walk-zone applicants than non-walk-zone applicants. Moreover, if there are more non-walk-zone applicants than walk-zone applicants, the outcomes will differ only for a small set of applicants who are admitted at the end of the process.

[8] The law of large numbers implies that this would be the expected outcome across repeated tie-breaker lottery realizations.

3.1 Decentralized Model

There is a finite set I of students and a school a with a finite set of slots S^a. Each slot $s \in S^a$ has a linear priority order π^s over students in I. The linear priority order π^s captures the "property rights" of the students for slot s, in the sense that the higher a student is ranked under π^s, the stronger claim he or she has for slot s of school a. The **total capacity** of a is $q_a \equiv |S^a|$.

We are interested in situations in which slot priorities are heterogeneous across slots. A consequence of such within-school heterogeneity is that we must determine how slots are assigned when a student is "qualified" for multiple slots that have different priority rankings. We suppose that the slots in S^a are ordered according to a (linear) **order of precedence** $\triangleright^a$. Given two slots $s, s' \in S^a$, the expression $s \triangleright^a s'$ means that slot s at school a is to be filled before slot s' whenever possible.

Given school a with set of slots S^a, profile of slot priorities $(\pi^s)_{s \in S^a}$, and order of precedence $\triangleright^a$ with

$$s_a^1 \triangleright^a s_a^2 \triangleright^a \cdots \triangleright^a s_a^{q_a},$$

the **choice of school *a* from set of students *J***, denoted by $C^a(J)$, is obtained as follows: slots at school a are filled one at a time following the order of precedence $\triangleright^a$. The highest-priority student in J under $\pi^{s_a^1}$, say student j_1, is chosen for slot s_a^1 of school a; the highest-priority student in $J \setminus \{j_1\}$ under $\pi^{s_a^2}$ is chosen for slot s_a^2, and so on.

We are particularly interested in slot priority structures in which some of the slots are reserved for applicants of a particular type (the "reserve-eligible"), while the remaining slots are open. Suppose there is a master priority order π^o that is uniform across all schools. This master priority is often determined by a random tie-breaker or by performance on an admissions exam (or in previous grades). For school a, there is a set $I_a \subseteq I$ of **reserve-eligible students**. Students who are not reserve-eligible are called **reserve-ineligible**. There are two types of slots:

1. Priorities at **open** slots correspond to the master priority order—$\pi^s = \pi^o$ for each open slot s.

2. Priorities at **reserve** slots grant all reserve-eligible students priority over all reserve-ineligible students, with the priority order within each group determined according to the master priority order π^o.

In Indian affirmative action systems, the reserve-eligible students are those from disadvantaged castes (Bagde, Epple, and Taylor 2016). Aygün and Bo (2013) have described reserves for public universities in Brazil, where the reserve-eligible are racial minorities, applicants from low income families, and applicants from public high schools. In Boston Public Schools, the reserve-eligible groups are students who live in the school's walk-zone, and thus, at times, we refer to BPS reserve slots as *walk-zone seats* (and refer to BPS open slots as *open seats*).[9]

[9]BPS also uses sibling priority, but for our theoretical analysis we consider a simplified priority structure that only depends on walk-zone status; using data from BPS, we show that this is a good approximation.

3.2 The Effects of Priority and Precedence Changes

We first examine the effects of increasing the reserve size given a precedence order. Suppose that slot s_* at school a is an open slot under priority profile π but is a reserve slot under priority profile $\tilde{\pi}$. Suppose that $\pi^s = \tilde{\pi}^s$ for all slots $s \neq s_*$. Let C^a and D^a respectively be the choice functions for a induced by the priorities π and $\tilde{\pi}$ under precedence order $\triangleright^a$. We obtain the following result.

Proposition 1. *Suppose that D^a is the choice function for school a obtained from C^a by changing an open slot to a reserve slot (fixing all other slots' priorities, as well as the precedence order). For any set of students $\bar{I} \subseteq I$:*

(i) All students that are reserve-eligible at school a and are chosen from $\bar{I}$ under choice function C^a are chosen under choice function D^a.

(ii) All students that are reserve-ineligible at school a and are chosen from $\bar{I}$ under choice function D^a are chosen under choice function C^a.

Proposition 1 states that when a school increases its reserve size, it admits weakly more reserve-eligible students and weakly fewer reserve-ineligible students. For Boston, this result suggests that increasing the walk-zone percentage beyond 50% may increase neighborhood assignment.

What is much less apparent, however, is that swapping the precedence order of a reserve slot and a subsequent open slot has the same qualitative effect as increasing the reserve size. Suppose now that s_r is a reserve slot of school a that immediately precedes an open slot s_o under the precedence order $\triangleright^a$. Suppose, moreover, that precedence order $\tilde{\triangleright}^a$ is obtained from $\triangleright^a$ by swapping the positions of s_r and s_o and leaving all other slot positions unchanged. Let C^a and D^a respectively be the choice functions for a induced by the precedence orders $\triangleright^a$ and $\tilde{\triangleright}^a$ under slot priorities π. We obtain the following analog to Proposition 1.

Proposition 2. *Suppose that D^a is the choice function for school a obtained from C^a by swapping the precedence of a reserve slot and a subsequent open slot (fixing all slot priorities, as well as all other precedence order positions). For any set of students $\bar{I} \subseteq I$:*

(i) All students that are reserve-eligible at school a and are chosen from $\bar{I}$ under choice function C^a are chosen under choice function D^a.

(ii) All students that are reserve-ineligible at school a and are chosen from $\bar{I}$ under choice function D^a are chosen under choice function C^a.

Together, Propositions 1 and 2 show how priority and precedence changes are substitute levers for influencing the assignment of reserve-eligible applicants. While the role of the number of reserve slots is quite apparent, the role of the order of precedence is much more subtle. Indeed, the choice of precedence order is often considered a minor technical detail, and, to our knowledge, precedence never explicitly entered school choice policy discussions until we raised the topic in Boston in parallel with the present work.

Qualitatively, the effect of decreasing a reserve slot's precedence order position is similar to the effect of replacing an open slot with a reserve slot. While this may initially appear counterintuitive, the reason is simple: decreasing the precedence of a reserve slot means a reserve-eligible student with high enough master priority to be eligible for both open and reserve slots may now be assigned to an open slot. This in turn increases competition for open slots and decreases competition for reserve slots.

Our observation about how changing applicant processing orders influences access for reserve-eligible applicants also surfaced in debates on affirmative policies in India. India's constitution stipulates that government-funded educational institutes and public sector jobs, including seats in parliament, hold reservations for disadvantaged groups. In 1975, a debate about applicant processing made its way to the Supreme Court, where a judge ruled that the "benefits of the reservation shall be snatched away by the top creamy layer of the backward class, thus leaving the weakest among the weak and leaving the fortunate layers to consume the whole cake."[10] In the context of our model, if reserve seats have higher precedence than open seats, then priority goes to applicants who do not need it (the "creamy layer"), leaving the remaining reserve-eligible students without opportunities to obtain open seats becauese they are out-competed by the reserve-ineligible students.

So far, our results are for a single school with a given choice function; this analysis directly informs us about reserves implemented in decentralized admissions in India and elsewhere. Since many centralized systems can be seen as iterated applications of choice functions, our results also yield an approximation for those centralized systems. We next formally examine how our results extend to centralized systems that use the deferred acceptance algorithm.

4 Centralized Admissions Systems with Reserves

Suppose now that there is a set of schools A. We use the notation a_0 to denote a "null school" representing the possibility of being unmatched; we assume this option is always available to all students. Let $S \equiv \bigcup_{a \in A} S^a$ denote the set of all slots (at nonnull schools). Each school $a \in A$ has a reserve-eligible population $I_a \subseteq I$, slot priorities $(\pi^s)_{s \in S^a}$, precedence order $\triangleright^a$, and choice function C^a, as described in the preceding section. Meanwhile, each student i has a strict preference relation P^i over $A \cup \{a_0\}$ (with associated weak preference relation R^i). If a_0 is preferable to $a \in A$ under P^i, then we say that a is **unacceptable to i**.

A **matching** $\mu : I \to A \cup \{a_0\}$ is a function that assigns each student to a school (or the null school) so that no school is assigned to more students than it has slots. This model generalizes the school choice model of Abdulkadiroğlu and Sönmez (2003) in that it allows for heterogeneous priorities across a given school's slots. Nevertheless, a mechanism based on the celebrated *(student-proposing) deferred acceptance algorithm* (Gale and Shapley 1962) easily extends to our model, given our earlier description of schools' choice functions.

[10]The court case is *State of Kerala vs. N. M. Thomas* (1974).

For a given profile of slot priorities $(\pi^s)_{s \in S}$ and an order of precedence $\triangleright^a$ for each school $a \in A$, the outcome of the **(student-proposing) deferred acceptance mechanism** can be obtained as follows:

***Step* 1.** Each student i applies to his/her most-preferred school in $A \cup \{a_0\}$. Each school $a \in A$ with a set of Step 1 applicants J_1^a tentatively holds the applicants in $C^a(J_1^a)$ and rejects the rest.[11]

***Step* ℓ.** Each student rejected in Step $\ell - 1$ applies to his/her most-preferred school in $A \cup \{a_0\}$ that has not yet rejected him/her. Each school $a \in A$ considers the set J_ℓ^a comprised of the new applicants to a and the students held by a at the end of Step $\ell - 1$, tentatively holds the applicants in $C^a(J_\ell^a)$, and rejects the rest.

The algorithm terminates after the first step in which no students are rejected, and assigns students to the schools holding their applications at that time.

4.1 Comparative Statics for Deferred Acceptance

In the context of deferred acceptance, we look at a single school and consider the effects of replacing open slots with reserve slots, and swapping the precedence order positions of reserve slots and lower-precedence open slots; we find that both changes weakly increase the number of reserve-eligible students assigned to the school.

Proposition 3. *Consider centralized assignment under (student-proposing) deferred acceptance, and fix a school $a \in A$.*

(i) Replacing an open slot of school a with a reserve slot (fixing all other slots' priorities, as well as all precedence orders) weakly increases the number of reserve-eligible students assigned to school a.

(ii) Switching the precedence order position of a reserve slot of school a with the position of a subsequent open slot (fixing all slot priorities, as well as all other precedence order positions) weakly increases the number of reserve-eligible students assigned to school a.

Proposition 3 is analogous to the results for decentralized admission for a single school in Propositions 1 and 2. However, Proposition 3's proof is substantially more involved, as it is necessary to consider how changes at one school cascade through the system under the centralized algorithm. That is, when either the precedence order or reserve size changes, a different set of applicants may apply to school a, thereby initiating a sequence of applications to other schools at subsequent stages of the deferred acceptance algorithm. These subsequent applications need to be tracked carefully. Indeed, neither comparative static result in Proposition 3 follows from earlier comparative static approaches used in simpler models (e.g., Balinski and Sönmez (1999)) because our comparative

[11] Here and in future steps, the null school a_0 always holds the full set of students that apply to it.

static involves simultaneous priority improvements for multiple students. As a result, we must develop a new proof strategy, which may be of independent interest in settings involving multi-agent priority improvements in matching models.

4.2 Aggregate Effects

Our analysis thus far has focused on assignment of reserve-eligible students to a particular school at which there is a change in reserve size or precedence. A natural question is whether increased reserve-eligible student assignment at a particular school always translates into increased overall reserve-eligible assignment across all schools. As usual in going from partial to general equilibrium analysis, the aggregate comparative static is not a foregone conclusion. In particular, it is well-known that in matching models, interactions across the market can lead to counterintuitive overall predictions. The following example shows that our results for a single school do not always imply an aggregate increase in reserve-eligible assignment.

Example 1. There are three schools, $A = \{k, l, m\}$. Schools k and m each have two slots and school l has three slots. There are seven students $I = \{i_1, i_2, i_3, i_4, i_5, i_6, i_7\}$. The reserve-eligible students are given by $I_k = \{i_1, i_7\}$, $I_l = \{i_2, i_3, i_4\}$, and $I_m = \{i_5, i_6\}$. The master priority π^o orders the students as

$$\pi^o : i_7 \succ i_2 \succ i_5 \succ i_3 \succ i_1 \succ i_6 \succ i_4.$$

The preference profile is:[12]

P^{i_1}	P^{i_2}	P^{i_3}	P^{i_4}	P^{i_5}	P^{i_6}	P^{i_7}
k	m	l	l	k	l	k
l			m	l	m	

First consider the case in which school k's first and school l's second slots are reserve slots, and all other slots are open slots. The outcome of deferred acceptance for this case is

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 \\ l & m & l & l & k & m & k \end{pmatrix}.$$ [13]

Observe that in addition to the two reserve slots assigned to reserve-eligible students i_4 and i_7, two of the open slots (namely those assigned to i_3 and i_6) are also assigned to reserve-eligible students. As such, four students are assigned to schools at which they are reserve-eligible.

Next, we replace the open slot at school k with a reserve slot, so that both slots at school k are reserve slots. We keep the slot sets and precedence orders of the other schools the same. The deferred acceptance outcome for the second case is

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 \\ k & m & l & m & l & l & k \end{pmatrix}.$$

[12] In our notation here and in future examples, preference relations are read vertically, and we omit the null school (as well as unacceptable schools) from the end of each preference relation, for notational simplicity. Thus, for example, P^{i_1} as stated means that i_1 prefers k to l and only finds schools k and l acceptable (i.e., preferable to a_0).

[13] Here and hereafter, we use this notation to indicate that i_1 is assigned to l, i_2 is assigned to m, and so forth.

Observe that while all three reserve slots are assigned to reserve-eligible students (students i_1, i_3, and i_7) none of the open slots are assigned to reserve-eligible students. That is, the total number of reserve-eligible student assignments decreases when the open slot at school k is replaced with a reserve slot. □

The preceding example illustrates that the direct "first-order" effect of a reserve change at a given school may be undone by the indirect effect on other schools. Moreover, it is easy to modify Example 1 to show that when the precedence order position of a reserve slot is swapped with that of a subsequent open slot, the overall reserve-eligible student assignment need not increase (see Example 2 in Appendix A). These negative findings highlight the complexity of distributional comparative statics in matching models with slot-specific priorities (see also Kominers and Sönmez (2013, 2016)).

4.3 Aggregate Effects Under Uniform Reserve Priority

One important feature of Example 1 is that the set of reserve-eligible students differs by school. When reserves represent walk-zone seats, as in Boston, we would expect reserves to differ by school since families are dispersed geographically (and thus live in different walk-zones). However, in a case like India, in which the reserve is intended to remedy a non-geographical disadvantage, i.e., membership in a particular caste, the set of reserve-eligible students is the same for each school.

If we have $I_a = I_{a'}$ for all pairs of schools $a, a' \in A$, then we say that we have **uniform reserve priority**.[14] In case of uniform reserve priority, it is still possible that reserve-eligible assignment can decrease when an open slot is replaced with a reserve slot (and likewise when a reserve slot is swapped with a subsequent open slot; see Example 3 in Appendix A). However, even in the worst-case scenario, only one fewer reserve-eligible student can be assigned under uniform reserve priority.

Proposition 4. *Consider centralized assignment under (student-proposing) deferred acceptance, and suppose that we have uniform reserve priority (that is, for any two schools $a, a' \in A$, we have $I_a = I_{a'}$). Then:*

(i) Replacing an open slot of a school with a reserve slot (fixing all other slots' priorities, as well as all precedence orders) cannot decrease the total assignment of reserve-eligible students across all schools by more than 1.

(ii) Switching the precedence order position of a reserve slot of a school with the position of a subsequent open slot (fixing all slot priorities, as well as all other precedence order positions) cannot decrease the total assignment of reserve-eligible students across all schools by more than 1.

[14] Aygün and Turhan (2016) have described a centralized admissions procedure with uniform reserve priority in the Indian state of Maharashtra.

In contrast to Proposition 4, if we do not have uniform reserve priority, then it is easy to show that when an open slot is replaced with a reserve slot, the number of reserve-eligible students assigned can be reduced by more than 1 (see Example 4 in Appendix A).

4.4 Aggregate Effects in the Two-School Case

When there are only two schools and there are enough slots for all the students, the effects of the reserve and precedence order changes described in Section 4.1 can be sharpened: either change weakly increases total reserve-eligible assignment. For the next result, we assume that there are only two schools, that each student is reserve-eligible at exactly one school, that there are enough slots for all the students, and that all students rank both schools.

Proposition 5. *Consider centralized assignment under (student-proposing) deferred acceptance, suppose that there are two (nonnull) schools, that each student is reserve-eligible at exactly one school, that there are enough slots for all the students, and that all students rank both schools. Then:*

(i) Replacing an open slot of either school with a reserve slot (fixing all other slots' priorities, as well as all precedence orders) weakly increases the total assignment of students to schools at which they are reserve-eligible.

(ii) Switching the precedence order position of a reserve slot of either school with the position of a subsequent open slot (fixing all slot priorities, as well as all other precedence order positions) weakly increases the total assignment of students to schools at which they are reserve-eligible.

(iii) The number of students assigned to their most-preferred schools is independent of both the number of reserve slots at each school and the precedence order profile.

The first two parts of Proposition 5 show that when there are only two schools, the aggregate effects of the reserve and precedence changes examined for a particular school in Proposition 3 extend across all schools. The last part of Proposition 5 shows that in the two-school case, changes in the reserve size or precedence orders are entirely distributional—both instruments leave the aggregate number of students obtaining their most-preferred schools unchanged.

Proposition 5 suggests a method to compute the policies with the minimum and maximum numbers of students assigned to schools at which they are reserve-eligible: In the two-school case, at least, the minimum (across all priority and precedence policies) is obtained when all slots are open slots; the maximum is obtained when all slots are reserve slots.

The analysis from the two-school case suggests that the outcomes illustrated in our negative examples require an elaborate sequence of applications and rejections involving more than two schools. We next turn to data from Boston and see that the results of Proposition 5 better approximate Boston's situation than what is suggested by our negative examples, which critically depend on carefully constructed rejection chains.

5 Precedence and Reserves in Boston

Table 2 reports the number of walk-zone students assigned to schools under different walk-zone percentages using the same tie-breaker lottery numbers as BPS used.[15] For each grade, more students are assigned to schools in their walk zones under 100% Walk than under 0% Walk. For Grade K1, the range between the two reserve policies is 11.2% of all students, which corresponds to 938 students. The range is 9.3% for Grade K2 and 5.4% for Grade 6. Consistent with Proposition 5, a higher reserve size corresponds to more walk-zone assignment.

The **Walk-Open** precedence order has all walk-zone seats precede open seats, while the **Open-Walk** precedence order has all open seats precede walk-zone seats. Consistent with Proposition 5, Table 2 shows that with a 50-50 seat split, more walk-zone students are assigned under Open-Walk precedence than under Walk-Open precedence. Moreover, the outcome of the Actual BPS policy is nearly identical to Walk-Open.[16] Table 2 also shows that the outcome of Walk-Open with 50% Walk is very close to 0% Walk, whereas the outcome of Open-Walk with 50% Walk is substantially different from 100% Walk. For Grade K1, the range between the two extremal precedence policies is 8.3%, or 691 students. This range corresponds to roughly three-quarters of the range between the two extremal reserve policies. For Grade K2, the precedence range is also three-quarters of the reserve range, while for Grade 6 it is about two-thirds. These empirical observations show that the maximal effect of changes in precedence policy is nearly as large as that corresponding to maximal changes in reserve size.

What policy would implement BPS's intended 50-50 compromise? To answer this question, it is worth returning to the example in Figure 1. In Panel (b) of Figure 1, walk-zone applicants who are rejected from the walk-zone half and then apply for open seats have systematically worse tie-breaker lottery numbers and are out-numbered by non-walk-zone students. This results in two biases: (1) the walk-zone students who remain have the least-favorable tie-breakers among walk-zone applicants, leaving them unlikely to be assigned ahead of non-walk-zone applicants, and (2) there are twice as many non-walk-zone applicants as walk-zone applicants in the residual pool for open seats. We refer to the first phenomenon as *random number bias* and the second as *processing bias.*

To examine the random number bias, Table 3 investigates the effects of using separate tie-breaker lotteries for the walk-zone and open seats. Column 2 reports on the Walk-Open precedence order with two tie-breaker lottery numbers. Even with two lotteries, there is processing bias, as the pool of walk-zone applicants is still depleted by the time the open seats are filled. Walk-Open with two lottery numbers assigns 48.4% of students to walk-zone schools at Grade K1 and is still close to the 46.6% assigned when Walk-Open is used with only one lottery number. That is, Walk-Open with two lottery numbers is much closer to 0% Walk than 100% Walk; this suggests that random number bias is only part of the reason Boston's assignment outcome is not midway between the 0% Walk and 100% Walk extremes.

[15] Appendix E provides details on the sample.

[16] Appendix E elaborates on the differences between Actual BPS and Open-Walk.

Although it eliminates the random number bias, the remedy of using two lottery numbers has an important drawback. It is well-known that using multiple lottery numbers across schools with the deferred acceptance algorithm may generate efficiency losses (Abdulkadiroğlu and Sönmez 2003, Abdulkadiroğlu, Pathak, and Roth 2009, Ashlagi, Nikzad, and Romm 2015). Even though the two lottery numbers are within schools (and not across schools), the same efficiency consequence arises here. The Unassigned row in the tables provides indirect evidence for this fact: Comparing Table 2 to Table 3, for each precedence policy under the 50-50 split, there are at least as many unassigned students and sometimes more with two tie-breaker lotteries.

Open-Walk eliminates both types of bias because neither the lottery number distribution nor the applicant pool is affected by application processing at the open half. (In the example illustrated in Figure 1, Open-Walk would result in 75 students from the walk zone being assigned.) Thus, distributional objectives may need to be accommodated by adjusting the reserve size.

To return to the Boston policy discussion, we conclude our investigation by examining how far the actual Boston system was from its intended 50-50 compromise. Table 4 computes the reserve-size adjustment, under Open-Walk, that corresponds to BPS's implementation of the 50-50 reserve. Depending on the grade, BPS's implementation corresponds to Open-Walk with roughly a 5-10% walk-zone reserve share. For Grade K1, the actual BPS implementation gives 47.2% of students walk-zone assignments; this is just above the Open-Walk treatment with a 5% walk-zone reserve (46.9%) but below the Open-Walk treatment with 10% walk-zone reserve (47.6%). For Grade K2, the actual BPS implementation has 48.5% walk-zone assignment, a figure close to the Open-Walk treatment with a 10% walk-zone reserve. For Grade 6, the actual BPS implementation is bracketed by Open-Walk with 5% and 10% walk-zone reserves. An unbiased version of the BPS implementation reveals it to be a substantial distance from the intended 50-50 compromise; indeed, the BPS implementation is closer to a 10-90 compromise.

6 Conclusion

Admissions policies in which applicants can be granted more than one type of seat raise questions about how seats should be processed. We have shown how both reserves and precedence are policy tools that have qualitatively similar impacts on school admission outcomes. We have also examined how those results generalize to centralized assignment systems.

Our analysis resolved a puzzle underlying a policy debate in Boston. Many groups in Boston felt that the BPS school assignment system did not sufficiently value children attending schools close to their homes despite the stated policy reserving half of each school's seats for walk-zone applicants. The resolution of this puzzle hinges on the important and surprising role played by Boston's chosen precedence order.

In addition to our comparative static results, our empirical analysis shows how the chosen precedence order effectively undermined the policy goal of the 50-50 seat split in Boston. Moreover, our empirical results establish that, in Boston, the precedence order (1) is an important lever for achieving distributional objectives and (2) has quantitative impacts of magnitudes similar to those of changes in reserve sizes.

The role of precedence order on admissions was not understood at the time of Boston's 50-50 compromise, and it is clear that Boston did not intend to choose a precedence order that undermined the walk-zone reserve (EAC 2013). When our work first made clear the unintended consequences of Boston's precedence choice, our findings were immediately of interest to all sides of the 2012-13 Boston school choice debate. Neighborhood schooling advocates were upset to learn that the precedence order had rendered the walk-zone reserve ineffective. School choice proponents, by contrast, pushed to either maintain the Walk-Open precedence order or eliminate the walk-zone reserve entirely. (For details on policy discussions and the impact of our research, see Appendix D.) Central to our own view was the need to encourage transparency: it is not sufficient to express the reserve policy without also specifying the precedence order.

Pursuant to our work, Boston Superintendent Dr. Carol Johnson (2013) proposed eliminating walk-zone priority entirely, as it had not been working as intended, and because the new choice menu system (Shi 2013) baked in a form of geographic preference under which students could only apply to schools close to their homes. The new BPS admissions policy took effect for placing elementary and middle school students in the 2013-14 school year.

Table 1. Differences between the Boston 50-50 Implementation and Alternative Walk-Zone Reserve Sizes

	Grade K1			Grade K2			Grade 6		
		Difference from BPS implementation			Difference from BPS implementation			Difference from BPS implementation	
	# Students	0% Walk	100% Walk	# Students	0% Walk	100% Walk	# Students	0% Walk	100% Walk
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
2009	1770	46	336	1715	28	343	2348	54	205
		3%	19%		2%	20%		2%	9%
2010	1977	68	392	1902	62	269	2308	41	171
		3%	20%		3%	14%		2%	7%
2011	2071	50	387	1821	90	293	2073	4	225
		2%	19%		5%	16%		0%	11%
2012	2515	88	504	2301	101	403	2057	24	247
		3%	20%		4%	18%		1%	12%
All	8333	252	1619	7739	281	1308	8786	123	848
		3%	19%		4%	17%		1%	10%

Notes. Table reports fraction of applicants whose assignments differ between Boston Public Schools' (BPS) 50-50 implementation and two alternative mechanisms, one without any walk-zone priority (0% Walk) and the other with walk-zone priority at all seats (100% Walk).

Table 2. Number of Students Assigned to Walk-Zone Schools, using One Lottery Number

	Priorities = 0% Walk	Priorities = 50% Walk Changing Precedence			Priorities = 100% Walk
		Walk-Open	Actual BPS	Open-Walk	
	(1)	(2)	(3)	(4)	(5)
			I. Grade K1		
Walk Zone	3849	3879	3930	4570	4787
	46.2%	46.6%	47.2%	54.8%	57.4%
Outside Walk Zone	2430	2399	2353	1695	1468
	29.2%	28.8%	28.2%	20.3%	17.6%
Unassigned	2054	2055	2050	2068	2078
	24.6%	24.7%	24.6%	24.8%	24.9%
			II. Grade K2		
Walk Zone	3651	3685	3753	4214	4374
	47.2%	47.6%	48.5%	54.5%	56.5%
Outside Walk Zone	2799	2764	2694	2214	2036
	36.2%	35.7%	34.8%	28.6%	26.3%
Unassigned	1289	1290	1292	1311	1329
	16.7%	16.7%	16.7%	16.9%	17.2%
			III. Grade 6		
Walk Zone	3439	3476	3484	3797	3907
	39.1%	39.6%	39.7%	43.2%	44.5%
Outside Walk Zone	4782	4750	4743	4419	4309
	54.4%	54.1%	54.0%	50.3%	49.0%
Unassigned	565	560	559	570	570
	6.4%	6.4%	6.4%	6.5%	6.5%

Notes. Table reports fraction of applicants assigned to walk-zone schools under several alternative assignment procedures, using data from 2009-2012. 0% Walk implements the student-proposing deferred acceptance mechanism with no walk-zone priority; 100% Walk implements the student-proposing deferred acceptance mechanism with all slots having walk-zone priority. Columns (2)-(4) hold the 50-50 school seat split fixed. Walk-Open implements the precedence order in which all walk-zone slots are ahead of all open slots. Actual BPS implements BPS's exact system (see Appendix E). Open-Walk implements the precedence order in which all open slots are ahead of all walk-zone slots.

Table 3. Number of Students Assigned to Walk-Zone Schools, using Two Lottery Numbers

	Priorities = 0% Walk	Priorities = 50% Walk Changing Precedence		Priorities = 100% Walk
		Walk-Open: Two Lotteries	Open-Walk: Two Lotteries	
	(1)	(2)	(3)	(4)
		I. Grade K1		
Walk Zone	3849	4034	4556	4787
	46.2%	48.4%	54.7%	57.4%
	61.3%	64.5%	72.7%	76.5%
Outside Walk Zone	2430	2217	1709	1468
	29.2%	26.6%	20.5%	17.6%
Unassigned	2054	2082	2068	2078
	24.6%	25.0%	24.8%	24.9%
		II. Grade K2		
Walk Zone	3651	3880	4210	4374
	47.2%	50.1%	54.4%	56.5%
Outside Walk Zone	2799	2539	2220	2036
	36.2%	32.8%	28.7%	26.3%
Unassigned	1289	1320	1309	1329
	16.7%	17.1%	16.9%	17.2%
		III. Grade 6		
Walk Zone	3439	3516	3784	3907
	39.1%	40.0%	43.1%	44.5%
Outside Walk Zone	4782	4655	4415	4309
	54.4%	53.0%	50.3%	49.0%
Unassigned	565	615	587	570
	6.4%	7.0%	6.7%	6.5%

Notes. Table reports fraction of applicants assigned to walk-zone schools under several alternative assignment procedures, using data from 2009-2012. 0% Walk implements the student-proposing deferred acceptance mechanism with no slots having walk-zone priority; 100% implements the student-proposing deferred acceptance mechanism with all slots having walk-zone priority. Columns (2)-(3) hold the 50-50 school seat split fixed. Walk-Open implements the precedence order in which all walk-zone slots are ahead of all open slots, but different lottery numbers are used for walk-zone and open slots. Open-Walk implements the precedence order in which all open slots are ahead of all walk-zone slots, but different lottery numbers are used for walk-zone and open slots. The same lottery numbers are used for each simulation.

Table 4. What Policy Was Being Implemented in Boston?

	Priorities = 0% Walk	Priorities = 5% Walk Open-Walk: One Lottery	Actual BPS	Priorities = 10% Walk Open-Walk: One Lottery
	(1)	(2)	(3)	(4)
		I. Grade K1		
Walk Zone	3849	3906	3930	3965
	46.2%	46.9%	47.2%	47.6%
Outside Walk Zone	2430	2369	2353	2304
	29.2%	28.4%	28.2%	27.6%
Unassigned	2054	2058	2050	2064
	24.6%	24.7%	24.6%	24.8%
		II. Grade K2		
Walk Zone	3651	3692	3753	3743
	47.2%	47.7%	48.5%	48.4%
Outside Walk Zone	2799	2757	2694	2702
	36.2%	35.6%	34.8%	34.9%
Unassigned	1289	1290	1292	1294
	16.7%	16.7%	16.7%	16.7%
		III. Grade 6		
Walk Zone	3439	3461	3484	3496
	39.1%	39.4%	39.7%	39.8%
Outside Walk Zone	4782	4751	4743	4715
	54.4%	54.1%	54.0%	53.7%
Unassigned	565	574	559	575
	6.4%	6.5%	6.4%	6.5%

Notes. Table reports fraction of applicants assigned to walk-zone schools under several alternative assignment procedures, using data from 2009-2012. 0% Walk implements the student-proposing deferred acceptance mechanism with no walk-zone priority. Open-Walk implements the precedence order in which all open slots are ahead of all walk-zone slots. The same lottery numbers are used for each simulation.

Appendix (Online)

A Examples Omitted from the Main Text

A.1 Aggregate Comparative Statics After Swapping the Precedence Order Positions of a Reserve Slot and a Subsequent Open Slot

Here, we give an analog of Example 1 that shows that swapping the precedence order positions of a reserve slot and a subsequent open slot at a given school need not always imply an aggregate increase in reserve-eligible assignment.

Example 2. We suppose that there are three schools, $A = \{k, l, m\}$. Schools k and m have two slots, and school l has three. There are seven students, $I = \{i_1, i_2, i_3, i_4, i_5, i_6, i_7\}$. Let $I_k = \{i_1, i_7\}$, $I_l = \{i_2, i_3, i_4\}$, and $I_m = \{i_5, i_6\}$. The master priority π^o orders the students as

$$\pi^o : i_7 \succ i_2 \succ i_5 \succ i_3 \succ i_1 \succ i_6 \succ i_4.$$

The preference profile is:[17]

P^{i_1}	P^{i_2}	P^{i_3}	P^{i_4}	P^{i_5}	P^{i_6}	P^{i_7}
k	m	l	l	k	l	k
l			m	l	m	

.

First, we consider the case in which school k's first and school l's second slots are reserve slots and all other slots are open.

The outcome of deferred acceptance for this case is

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 \\ l & m & l & l & k & m & k \end{pmatrix}.$$

Observe that, in addition to the two reserve slots assigned to reserve-eligible students i_4 and i_7, two of the open slots (namely those assigned to students i_3 and i_6) are also assigned to reserve-eligible students. As such, four students are assigned to schools at which they are reserve-eligible.

Next, we change the order of precedence at school k so that its open slot has higher precedence than its reserve slot. We keep the slot priorities, as well as the precedence orders of the other schools unchanged.

The outcome of deferred acceptance for the second case is

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 \\ k & m & l & m & l & l & k \end{pmatrix}.$$

Observe that, while the two reserve slots are assigned to reserve-eligible students (here, students i_1 and i_3), only one of the open slots is assigned to reserve-eligible student (i_7).

That is, the total number of reserve-eligible student assignments decreases when the reserve slot at school k is swapped with a lower-precedence open slot. □

[17] Just as in the main text, in our notation here and in future examples, preference relations are read vertically, and we omit the null school (as well as unacceptable schools) from the end of each preference relation (see Footnote 12).

A.2 The Case of Uniform Reserve Priority

Now, we provide an example showing that *even under uniform reserve priority*, neither switching open slots to reserve slots nor swapping the precedence order positions of reserve slots with subsequent open slots need always imply an aggregate increase in reserve-eligible assignment.

Example 3. Suppose that there are three schools, $A = \{a, b, c\}$. Schools a and b have two slots, and school c has one. There are six students, $I = \{i_1, i_2, i_3, i_4, i_5, i_6\}$. The reserve-eligible students are $I_a = I_b = I_c = \{i_1, i_2, i_3, i_4\}$; there is uniform reserve priority, and students i_5 and i_6 are completely reserve-ineligible. The master priority π^o orders the students as

$$\pi^o : i_1 \succ i_5 \succ i_2 \succ i_3 \succ i_6 \succ i_4.$$

The preference profile is:

P^{i_1}	P^{i_2}	P^{i_3}	P^{i_4}	P^{i_5}	P^{i_6}
a	a	b	b	a	c
b	c	c	a	b	a
c	b	a	c	c	b

.

First, consider the case in which school a's first slot is a reserve slot and a's second slot is open, school b's first slot is open and second slot is reserve, and school c's slot is open. The outcome of deferred acceptance is

$$\mu_1 = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 \\ a & c & b & b & a & a_0 \end{pmatrix}.$$

Next, suppose that school a's second slot is a reserve slot. The outcome of deferred acceptance is

$$\mu_2 = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 \\ a & a & b & a_0 & b & c \end{pmatrix}.$$

Alternatively, suppose that school a's first slot is open, and second slot is reserve; this corresponds to a swap of the precedence order positions of the two slots. The outcome of deferred acceptance is

$$\mu_3 = \begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 \\ a & a & b & a_0 & b & c \end{pmatrix} = \mu_2.$$

Observe that under both μ_2 and μ_3, there is one fewer reserve-eligible student assigned than under μ_1 (student i_4). Thus, we see that total reserve-eligible assignment can decrease—even under uniform reserve priority. □

Note that in Example 3, total reserve-eligible assignment decreases by no more than 1 under both μ_2 and μ_3 (relative to μ_1)—as Proposition 4 implies should be the case. Without uniform reserve priority, however, total reserve-eligible assignment can decrease by more than 1 when an open slot is replaced by a reserve slot, as the following example shows.

Example 4. We modify the setting of Example 3, by adding two more schools, d and e, each with one slot, and two more students, i_7 and i_8. Suppose that we have $I_a = \{i_1, i_2\}$, $I_b = \{i_3, i_4\}$, $I_c = \{i_5, i_8\}$, $I_d = \{i_6\}$, and $I_e = \{i_7\}$, and

$$\pi^o : i_1 \succ i_5 \succ i_2 \succ i_3 \succ i_6 \succ i_7 \succ i_8 \succ i_4.$$

We modify the preferences of student i_6 and specify the preferences of students i_7 and i_8 as follows:[18]

P^{i_6}	P^{i_7}	P^{i_8}
c	d	e
a	e	
b		
d		
e		

.

First, we suppose that school a's first slot and school b's second slot are reserve slots, and all other slots are open. The outcome of deferred acceptance is

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 & i_8 \\ a & c & b & b & a & d & e & a_0 \end{pmatrix},$$

and five students are assigned to schools at which they are reserve-eligible.

If we replace the open slot at school a with a reserve slot, so that both slots at school a are reserve slots (holding all the other schools' slots and precedence orders fixed), the deferred acceptance outcome is

$$\begin{pmatrix} i_1 & i_2 & i_3 & i_4 & i_5 & i_6 & i_7 & i_8 \\ a & a & b & a_0 & b & c & d & e \end{pmatrix},$$

under which three students are assigned to schools at which they are reserve-eligible. $\square$

B Proofs of Propositions

B.1 Proof of Proposition 1 and Preliminaries for the Proof of Proposition 3.(i)

We prove the following lemma that implies Proposition 1; this lemma is then used in the proof of Proposition 3.(i).

We recall the setup of Proposition 1: slot s_* of school a is an open slot under priorities π but is a reserve slot under priorities $\tilde{\pi}$, and $\pi^s = \tilde{\pi}^s$ for all slots $s \neq s_*$; C^a and D^a are respectively the choice functions for a induced by the priorities π and $\tilde{\pi}$ under precedence order $\triangleright^a$.

Lemma 1. *For any set of students $\bar{I} \subseteq I$, as pictured in Figure 2:*

1. *All students that are reserve-eligible at a and are chosen from $\bar{I}$ under choice function C^a are chosen under choice function D^a (i.e., $[(C^a(\bar{I})) \cap I_a] \subseteq [(D^a(\bar{I})) \cap I_a]$). Moreover,*

$$|[(D^a(\bar{I})) \cap I_a] \setminus [(C^a(\bar{I})) \cap I_a]| \leq 1.$$

[18] We assume that d and e are unacceptable to students other than i_6, i_7, and i_8.

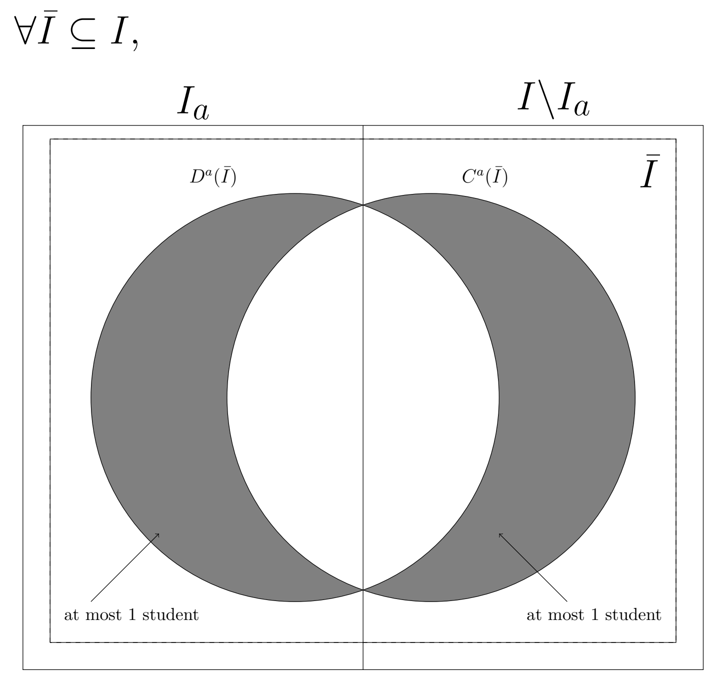


Figure 2: Comparison of $C^a(\bar{I})$ and $D^a(\bar{I})$, as described formally in the lemma.

2. *All students that are reserve-ineligible at a and are chosen from $\bar{I}$ under choice function D^a are chosen under choice function C^a (i.e., $[(D^a(\bar{I}))\cap(I\setminus I_a)] \subseteq [(C^a(\bar{I}))\cap(I\setminus I_a)]$). Moreover,*

$$|[(C^a(\bar{I})) \cap (I \setminus I_a)] \setminus [(D^a(\bar{I})) \cap (I \setminus I_a)]| \leq 1.$$

Proof. We proceed by induction on the number q_a of slots at a. The base case $q_a = 1$ is immediate, as then $S^a = \{s_*\}$ and $C^a(\bar{I}) \neq D^a(\bar{I})$ if and only if a student that is reserve-eligible at a is assigned to s_* under D^a, but a reserve-ineligible student is assigned to s_* under C^a, that is, if $D^a(\bar{I}) \subseteq I_a$ while $C^a(\bar{I}) \subseteq (I \setminus I_a)$. It then follows immediately from the preceding observation that $[(C^a(\bar{I})) \cap I_a] \subseteq [(D^a(\bar{I})) \cap I_a]$ and $[(D^a(\bar{I})) \cap (I \setminus I_a)] \subseteq [(C^a(\bar{I})) \cap (I \setminus I_a)]$; we then see that $|[(D^a(\bar{I}))\cap I_a]\setminus[(C^a(\bar{I}))\cap I_a]| \leq 1$ and $|[(C^a(\bar{I}))\cap(I\setminus I_a)]\setminus[(D^a(\bar{I}))\cap(I\setminus I_a)]| \leq 1$, as $q_a = |S^a| = 1$.

Now, given the result for the base case $q_a = 1$, we suppose that the result holds for all $q_a < \ell$ for some $\ell > 1$; we show that this implies the result for $q_a = \ell$. We suppose that $q_a = \ell$, and let $\bar{s} \in S^a$ be the slot that is minimal (i.e., processed/filled last) under the precedence order $\rhd^a$. First, we note that as the choice function of a always assigns as many applicants as possible given the available slots, when choosing students from $\bar{I}$, a fills all its slots under C^a if and only if it fills all its slots under D^a; hence $\bar{s}$ is either full in both cases or empty in both cases. Moreover, the result follows directly from the inductive hypothesis in the case that no student is assigned to $\bar{s}$ (under either priority structure); hence, we assume that

$$|C^a(\bar{I})| = |D^a(\bar{I})| = q_a = \ell \leq |\bar{I}|. \tag{1}$$

If $\bar{s} = s_*$, then the result follows just as in the base case: It is clear from the algorithms defining C^a and D^a that in the computations of $C^a(\bar{I})$ and $D^a(\bar{I})$, the same students are assigned to slots s with higher precedence than $s_* = \bar{s}$ (i.e., slots s with $s \rhd^a s_* = \bar{s}$), as those slots' priorities and relative precedence ordering are fixed. Thus, as in the base case, $C^a(\bar{I}) \neq D^a(\bar{I})$ if and only if a student that is reserve-eligible at a is assigned to s_* under D^a, but a reserve-ineligible student is assigned to s_* under C^a.

If $\bar{s} \neq s_*$, then $s_* \ \rhd^a \ \bar{s}$. We let $J \subseteq \bar{I}$ be the set of students assigned to slots in $S^a \setminus \{\bar{s}\}$ in the computation of $C^a(\bar{I})$, and let $K \subseteq \bar{I}$ be the set of students assigned to slots in $S^a \setminus \{\bar{s}\}$ in the computation of $D^a(\bar{I})$.

The first $q_a - 1 = \ell - 1$ slots of a can be treated as a school with slot set $S^a \setminus \{\bar{s}\}$ (under the precedence order induced by $\rhd^a$). Thus, the inductive hypothesis (in the case $\ell - 1$) implies:

$$[J \cap I_a] \subseteq [K \cap I_a]; \tag{2}$$
$$|[K \cap I_a] \setminus [J \cap I_a]| \leq 1; \tag{3}$$
$$[K \cap (I \setminus I_a)] \subseteq [J \cap (I \setminus I_a)]; \tag{4}$$
$$|[J \cap (I \setminus I_a)] \setminus [K \cap (I \setminus I_a)]| \leq 1. \tag{5}$$

If we have equality in (2) and (4),[19] then the set of students available to be assigned to $\bar{s}$ in the computation of $C^a(\bar{I})$ is the same as in the computation of $D^a(\bar{I})$. Thus, as $\pi^{\bar{s}} = \tilde{\pi}^{\bar{s}}$ by assumption (recall that $\bar{s} \neq s_*$), we have $C^a(\bar{I}) = D^a(\bar{I})$; hence, the desired result follows immediately.[20]

If instead the inclusions in (2) and (4) are strict, then by (3) and (5), respectively, we have a unique student $k \in ([K \cap I_a] \setminus [J \cap I_a])$ and a unique student $j \in ([J \cap (I \setminus I_a)] \setminus [K \cap (I \setminus I_a)])$. Here, k is reserve-eligible at a and is assigned to a slot s with higher precedence than $\bar{s}$ (i.e., a slot s with $s \rhd^a \bar{s}$) in the computation of $D^a(\bar{I})$ but is not assigned to such a slot in the computation of $C^a(\bar{I})$. Likewise, j is reserve-ineligible at a, is assigned to a slot s with $s \rhd^a \bar{s}$ in the computation of $C^a(\bar{I})$, and is not assigned to such a slot in the computation of $D^a(\bar{I})$. By construction, k must be the π^o-maximal student in $[\bar{I} \setminus J] \cap I_a$ and j must be the π^o-maximal student in $[\bar{I} \setminus K] \cap (I \setminus I_a)$ (indeed, j is π^o-maximal in $\bar{I} \setminus K$).

Now:

- If $\bar{s}$ is a reserve slot, then k is assigned to $\bar{s}$ in the computation of $C^a(\bar{I})$; hence, $C^a(\bar{I}) = J \cup \{k\}$. Thus, as $k \in [K \cap I_a]$, we have $[(C^a(\bar{I})) \cap I_a] \subseteq [(D^a(\bar{I})) \cap I_a]$ by (2), and $|[(D^a(\bar{I})) \cap I_a] \setminus [(C^a(\bar{I})) \cap I_a]| \leq 1$ by (3). In the computation of $D^a(\bar{I})$, meanwhile, if a reserve-eligible student is not assigned to $\bar{s}$, then j must be assigned to $\bar{s}$, as j is π^o-maximal among students in $[\bar{I} \setminus K] \cap (I \setminus I_a)$. It follows that $[(D^a(\bar{I})) \cap (I \setminus I_a)] \subseteq [(C^a(\bar{I})) \cap (I \setminus I_a)]$ (by (4)), and $|[(C^a(\bar{I})) \cap (I \setminus I_a)] \setminus [(D^a(\bar{I})) \cap (I \setminus I_a)]| \leq 1$ (by (5)).

[19] As $|J| = |K|$ by (1), equality holds in one of (2) and (4) if and only if it holds for *both* inclusions (2) and (4).
[20] Note that when $C^a(\bar{I}) = D^a(\bar{I})$, we have

$$|[(D^a(\bar{I})) \cap I_a] \setminus [(C^a(\bar{I})) \cap I_a]| = 0 \leq 1 \text{ and } |[(C^a(\bar{I})) \cap (I \setminus I_a)] \setminus [(D^a(\bar{I})) \cap (I \setminus I_a)]| = 0 \leq 1.$$

- If $\bar{s}$ is an open slot, then j is assigned to $\bar{s}$ in the computation of $D^a(\bar{I})$; hence, $D^a(\bar{I}) = K \cup \{j\}$. Thus, as $j \in [J \cap (I \setminus I_a)]$, we have $[(D^a(\bar{I})) \cap (I \setminus I_a)] \subseteq [(C^a(\bar{I})) \cap (I \setminus I_a)]$ by (4), and $|[(C^a(\bar{I})) \cap (I \setminus I_a)] \setminus [(D^a(\bar{I})) \cap (I \setminus I_a)]| \leq 1$ by (5). Meanwhile, if a reserve-eligible student is assigned to $\bar{s}$ in the computation of $C^a(\bar{I})$, then it must be k, as k is π^o-maximal among students in $[\bar{I} \setminus J] \cap I_a$. It follows that $[(C^a(\bar{I})) \cap I_a] \subseteq [(D^a(\bar{I})) \cap I_a]$ (by (2)), and $|[(D^a(\bar{I})) \cap I_a] \setminus [(C^a(\bar{I})) \cap I_a]| \leq 1$ (by (3)).

These observations complete the induction. □

B.2 Proof of Proposition 2 and Preliminaries for the Proof of Proposition 3.(ii)

We prove the following lemma that implies Proposition 2; this lemma is then used in the proof of Proposition 3.(ii).

We recall the setup of Proposition 2: slot s_r is a reserve slot of a that immediately precedes an open slot s_o under the precedence order $\triangleright^a$; precedence order $\tilde{\triangleright}^a$ is obtained from $\triangleright^a$ by swapping the positions of s_r and s_o and leaving all other slot positions unchanged; C^a and D^a are respectively the choice functions for a induced by the precedence orders $\triangleright^a$ and $\tilde{\triangleright}^a$ under slot priorities π.

Lemma 2. *For any set of students $\bar{I} \subseteq I$, as pictured in Figure 2:*

1. *All students that are reserve-eligible at a and are chosen from $\bar{I}$ under choice function C^a are chosen under choice function D^a (i.e., $[(C^a(\bar{I})) \cap I_a] \subseteq [(D^a(\bar{I})) \cap I_a]$). Moreover,*
$$|[(D^a(\bar{I})) \cap I_a] \setminus [(C^a(\bar{I})) \cap I_a]| \leq 1.$$
2. *All students that are reserve-ineligible at a and are chosen from $\bar{I}$ under choice function D^a are chosen under choice function C^a (i.e., $[(D^a(\bar{I})) \cap (I \setminus I_a)] \subseteq [(C^a(\bar{I})) \cap (I \setminus I_a)]$). Moreover,*
$$|[(C^a(\bar{I})) \cap (I \setminus I_a)] \setminus [(D^a(\bar{I})) \cap (I \setminus I_a)]| \leq 1.$$

Proof. We proceed by induction on the number q_a of slots at a.

First, we prove the base case $q_a = 2$.[21] We denote by i_{s_r} and i_{s_o} (resp. j_{s_r} and j_{s_o}) the students respectively assigned to slots s_r and s_o in the computation of $C^a(\bar{I})$ (resp. $D^a(\bar{I})$).

Now:

- If $\{i_{s_r}, i_{s_o}\} \subseteq I_a$, then the ordering under π^o must rank i_{s_r} highest among all students in $\bar{I}$, and rank i_{s_o} second-highest among all students in $\bar{I}$, as otherwise some student $i \in \bar{I}$ with $i \neq i_{s_r}$ would have higher rank than i_{s_o} under π^o, and would thus have higher claim than i_{s_o} for (open) slot s_o under precedence order $\triangleright^a$. But then, i_{s_r} is the π^o-maximal student in $\bar{I}$ and i_{s_o} is the π^o-maximal reserve-eligible student in $\bar{I} \setminus \{i_{s_r}\}$; hence, we must have $j_{s_o} = i_{s_r}$ and $j_{s_r} = i_{s_o}$, so that $D^a(\bar{I}) = C^a(\bar{I})$. In this case, $|[(C^a(\bar{I})) \cap (I \setminus I_a)] \setminus [(D^a(\bar{I})) \cap (I \setminus I_a)]| = 0 \leq 1$ and $|[(D^a(\bar{I})) \cap I_a] \setminus [(C^a(\bar{I})) \cap I_a]| = 0 \leq 1$.

[21] Note that the setup requires at least two distinct slots of a, so $q_a \geq 2$ *a priori*.

- If $\{i_{s_r}, i_{s_o}\} \subseteq (I \setminus I_a)$, then $\bar{I}$ contains no students that are reserve-eligible at a (i.e., $\bar{I} \cap I_a = \emptyset$) and i_{s_r} and i_{s_o} are then just the π^o-maximal reserve-ineligible students in $\bar{I}$. In this case, we find that $j_{s_o} = i_{s_r}$ and $j_{s_r} = i_{s_o}$; hence, $D^a(\bar{I}) = C^a(\bar{I})$. Once again, we have $|[(C^a(\bar{I})) \cap (I \setminus I_a)] \setminus [(D^a(\bar{I})) \cap (I \setminus I_a)]| = 0 \leq 1$ and $|[(D^a(\bar{I})) \cap I_a] \setminus [(C^a(\bar{I})) \cap I_a]| = 0 \leq 1$.

- If $i_{s_r} \in I_a$ and $i_{s_o} \in (I \setminus I_a)$, then i_{s_r} is the π^o-maximal reserve-eligible student at a in $\bar{I}$. If i_{s_r} is also π^o-maximal among *all* students in $\bar{I}$, then we have $j_{s_o} = i_{s_r}$. Moreover, in this case either

  1. $j_{s_r} \in I_a$, or
  2. i_{s_r} is the only reserve-eligible student of a in $\bar{I}$, so that $j_{s_r} = i_{s_o}$.

  Alternatively, if i_{s_r} is not π^o-maximal among all students in $\bar{I}$, then i_{s_o} must be π^o-maximal among all students in I_a, so that $j_{s_o} = i_{s_o}$ and $j_{s_r} = i_{s_r}$.

  We therefore find that

  $$[(C^a(\bar{I})) \cap I_a] = \{i_{s_r}\} \subseteq [(D^a(\bar{I})) \cap I_a];$$

  hence, $|[(D^a(\bar{I})) \cap I_a] \setminus [(C^a(\bar{I})) \cap I_a]| \leq 1$, as $|(D^a(\bar{I})) \cap I_a| \leq q_a = 2$. Additionally, we have

  $$[(D^a(\bar{I})) \cap (I \setminus I_a)] \subseteq \{i_{s_o}\} = [(C^a(\bar{I})) \cap (I \setminus I_a)],$$

  so that $|[(C^a(\bar{I})) \cap (I \setminus I_a)] \setminus [(D^a(\bar{I})) \cap (I \setminus I_a)]| \leq 1$ (again, as $q_a = 2$).

- We cannot have $i_{s_r} \in (I \setminus I_a)$ and $i_{s_o} \in I_a$, as $s_r \rhd^a s_o$ and s_r is a reserve slot (and thus gives all students in I_a higher priority than students in $I \setminus I_a$).

The preceding four cases are exhaustive and the desired result holds in each; thus, we have the base case for our induction.

Now, given the result for the base case $q_a = 2$, we suppose that the result holds for all $q_a < \ell$ for some $\ell > 2$; we show that this implies the result for $q_a = \ell$. Thus, we suppose that $q_a = \ell$. First, we note that as the choice function of a always assigns as many applicants as possible given the available slots, when choosing students from $\bar{I}$, a fills all its slots under C^a if and only if it fills all its slots under D^a; equivalently, the slots that are lowest-precedence under $\rhd$ and $\tilde{\rhd}$ (respectively) are either full in both cases or empty in both cases. Moreover, the result follows directly from the inductive hypothesis in the case that no student is assigned to the slots that are lowest-precedence under $\rhd$ and $\tilde{\rhd}$; hence, we assume that

$$|C^a(\bar{I})| = |D^a(\bar{I})| = q_a = \ell \leq |\bar{I}|. \tag{6}$$

We let $\bar{s} \in S^a$ be the slot that is minimal under the precedence order $\rhd^a$. If $\bar{s} = s_o$, then the result follows just as in the base case, as it is clear from the procedures defining C^a and D^a that the same students are assigned to slots s with higher precedence than s_r under $\rhd^a$ (i.e., slots s with $s \rhd^a s_r \rhd^a s_o = \bar{s}$ and $s \,\tilde{\rhd}^a\, s_o \,\tilde{\rhd}^a\, s_r$) in the computations of $C^a(\bar{I})$ and $D^a(\bar{I})$.

If $\bar{s} \neq s_o$, then $s_r \rhd^a s_o \rhd^a \bar{s}$. We let $J \subseteq \bar{I}$ be the set of students assigned to slots in $S^a \setminus \{\bar{s}\}$ in the computation of $C^a(\bar{I})$, and let $K \subseteq \bar{I}$ be the set of students assigned to slots in $S^a \setminus \{\bar{s}\}$ in the computation of $D^a(\bar{I})$. The first $\ell - 1$ slots of a can be treated as a school with slot set $S^a \setminus \{\bar{s}\}$ (under the precedence order induced by $\rhd^a$). Thus, the inductive hypothesis (in the case $q_a = \ell - 1$), implies:

$$[J \cap I_a] \subseteq [K \cap I_a]; \tag{7}$$

$$|[K \cap I_a] \setminus [J \cap I_a]| \leq 1; \tag{8}$$

$$[K \cap (I \setminus I_a)] \subseteq [J \cap (I \setminus I_a)]; \tag{9}$$

$$|[J \cap (I \setminus I_a)] \setminus [K \cap (I \setminus I_a)]| \leq 1. \tag{10}$$

If we have equality in (7) and (9),[22] then the set of students available to be assigned to $\bar{s}$ in the computation of $C^a(\bar{I})$ is the same as in the computation of $D^a(\bar{I})$. Thus, we have $C^a(\bar{I}) = D^a(\bar{I})$; hence, the desired result follows immediately.[23]

If instead the inclusions in (7) and (9) are strict, then by (8) and (10), respectively, we have a unique student $k \in ([K \cap I_a] \setminus [J \cap I_a])$ and a unique student $j \in ([J \cap (I \setminus I_a)] \setminus [K \cap (I \setminus I_a)])$. Here, k is reserve-eligible at a and is assigned to a slot s with higher precedence than $\bar{s}$ (i.e., a slot s with $s \rhd^a \bar{s}$ and $s \tilde{\rhd}^a \bar{s}$) in the computation of $D^a(\bar{I})$ but is not assigned to such a slot in the computation of $C^a(\bar{I})$. Likewise, j is reserve-ineligible at a, is assigned to a slot s with higher precedence than $\bar{s}$ (i.e., a slot s with $s \rhd^a \bar{s}$ and $s \tilde{\rhd}^a \bar{s}$) in the computation of $C^a(\bar{I})$, and is not assigned to such a slot in the computation of $D^a(\bar{I})$. By construction, k must be the π^o-maximal student in $[\bar{I} \setminus J] \cap I_a$ and j must be the π^o-maximal student in $[\bar{I} \setminus K] \cap (I \setminus I_a)$ (indeed, j is π^o-maximal in $\bar{I} \setminus K$).

Now:

- If $\bar{s}$ is a reserve slot, then k is assigned to $\bar{s}$ in the computation of $C^a(\bar{I})$; hence, $C^a(\bar{I}) = J \cup \{k\}$. Thus, as $k \in [K \cap I_a]$, we have $[(C^a(\bar{I})) \cap I_a] \subseteq [(D^a(\bar{I})) \cap I_a]$ by (7), and $|[(D^a(\bar{I})) \cap I_a] \setminus [(C^a(\bar{I})) \cap I_a]| \leq 1$ by (8). In the computation of $D^a(\bar{I})$, meanwhile, if a reserve-eligible student is not assigned to $\bar{s}$, then j must be assigned to $\bar{s}$, as j is π^o-maximal among students in $[\bar{I} \setminus K] \cap (I \setminus I_a)$. It follows that $[(D^a(\bar{I})) \cap (I \setminus I_a)] \subseteq [(C^a(\bar{I})) \cap (I \setminus I_a)]$ (by (9)), and $|[(C^a(\bar{I})) \cap (I \setminus I_a)] \setminus [(D^a(\bar{I})) \cap (I \setminus I_a)]| \leq 1$ (by (10)).

- If $\bar{s}$ is an open slot, then j is assigned to $\bar{s}$ in the computation of $D^a(\bar{I})$; hence, $D^a(\bar{I}) = K \cup \{j\}$. Thus, as $j \in [J \cap (I \setminus I_a)]$, we have $[(D^a(\bar{I})) \cap (I \setminus I_a)] \subseteq [(C^a(\bar{I})) \cap (I \setminus I_a)]$ by (9), and $|[(C^a(\bar{I})) \cap (I \setminus I_a)] \setminus [(D^a(\bar{I})) \cap (I \setminus I_a)]| \leq 1$ by (10). Meanwhile, if a reserve-eligible student is assigned to $\bar{s}$ in the computation of $C^a(\bar{I})$, then it must be k, as k is π^o-maximal among students in $[\bar{I} \setminus J] \cap I_a$. It follows that $[(C^a(\bar{I})) \cap I_a] \subseteq [(D^a(\bar{I})) \cap I_a]$ (by (7)), and $|[(D^a(\bar{I})) \cap I_a] \setminus [(C^a(\bar{I})) \cap I_a]| \leq 1$ (by (8)).

[22] As $|J| = |K|$ by (6), equality holds in one of (7) or (9) if and only if it holds for *both* (7) and (9).

[23] Note that when $C^a(\bar{I}) = D^a(\bar{I})$, we have

$$|[(D^a(\bar{I})) \cap I_a] \setminus [(C^a(\bar{I})) \cap I_a]| = 0 \leq 1 \text{ and } |[(C^a(\bar{I})) \cap (I \setminus I_a)] \setminus [(D^a(\bar{I})) \cap (I \setminus I_a)]| = 0 \leq 1.$$

These observations complete the induction. □

B.3 Proof of Proposition 3

In this section, we prove Proposition 3.(i) and 3.(ii) using a completely parallel argument for the two results. Our proof makes use of two technical machinery components. The first, which is well-known in the matching literature, is the *cumulative offer process* (Kelso and Crawford 1982, Hatfield and Milgrom 2005, Hatfield and Kojima 2010, Hatfield, Kominers, and Westkamp 2016), a stable matching algorithm that is outcome-equivalent to deferred acceptance but easier to analyze in our framework. The second component, which to the best of our knowledge is completely novel, is the construction of a *copy economy*, a setting in which some students i who are rejected by school a are replaced by two "copies"—a top copy i^{t} who takes the role of i in applying to schools that i weakly prefers to a, and a bottom copy i^{b} who takes the role of i in applying to schools that i likes less than a.

Because bottom copies i^{b} can act independently of top copies i^{t} and the cumulative offer process is independent of proposal order, constructing copies enables us to track how the market responds to rejection of students i by a *before* i *(or rather,* i^{t}*) applies to* a.

B.3.1 The Cumulative Offer Process

Definition. In the **cumulative offer process** under choice functions $\bar{C}$, students propose to schools in a sequence of steps $\ell = 1, 2, \ldots$:

***Step* 1.** Some student $i^1 \in I$ proposes to his/her favorite school $a^1 \in (A \cup \{a_0\})$. Set $\bar{A}^2_{a^1} = \{i^1\}$, and set $\bar{A}^2_{a'} = \emptyset$ for each $a' \neq a^1$; these are the sets of students **available** to schools at the beginning of Step 2. Each school $b \in A$ **holds** $\bar{C}^b(\bar{A}^2_b)$ and **rejects** all other students in $\bar{A}^2_b$; the null school a_0 **holds** the full set $\bar{A}^2_{a_0}$ of students that have proposed to it.

***Step* ℓ.** Some student $i^\ell \in I$ not currently held by any school proposes to his/her most-preferred school that has not yet rejected him/her, $a^\ell \in (A \cup \{a_0\})$. Set $\bar{A}^{\ell+1}_{a^\ell} = \bar{A}^\ell_{a^\ell} \cup \{i^\ell\}$, and set $\bar{A}^{\ell+1}_{a'} = \bar{A}^\ell_{a'}$ for each $a' \neq a^\ell$. Each school $b \in A$ holds $\bar{C}^b(\bar{A}^{\ell+1}_b)$ and rejects all other students in $\bar{A}^{\ell+1}_b$; the null school a_0 holds the full set $\bar{A}^{\ell+1}_{a_0}$ of students that have proposed to it.

If at any Step $\ell + 1$ no student is able to propose—that is, if all students are held by schools (potentially including the null school a_0)—then the process terminates. The **outcome of the cumulative offer process** is the matching $\bar{\mu}$ that assigns each school $b \in (A \cup \{a_0\})$ the students it holds at the end of the last step before termination.[24]

In our context, the cumulative offer process outcome is independent of the proposal order and is equal to the outcome of the student-proposing deferred acceptance algorithm (see Hatfield and

[24] For all nonnull schools $b \in A$, we have $\bar{\mu}_b = \bar{C}^b(\bar{A}^{\ell+1}_b)$; for the null school a_0 we have $\bar{\mu}_{a_0} = \bar{A}^{\ell+1}_{a_0}$.

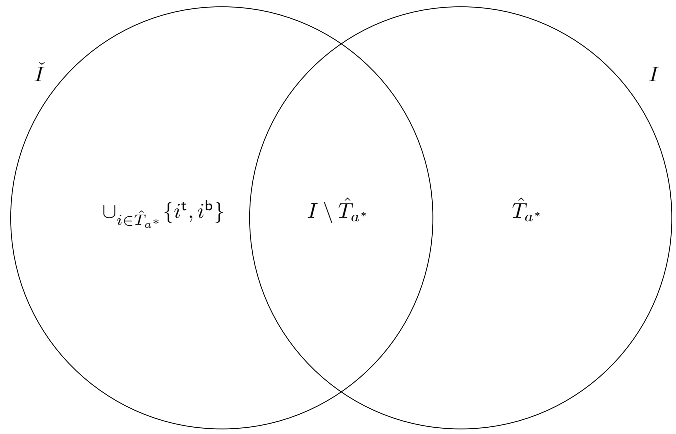


Figure 3: Relationship between the set $\check{I}$ of students in the copy economy and the set I of students in the original economy.

Kojima (2010), Kominers and Sönmez (2013), Hirata and Kasuya (2014)).[25]

B.3.2 Copy Economies

We denote by $\bar{\mu}$ the outcome of cumulative offer process under choice functions $\bar{C}$ (associated to priorities $\bar{\pi}$ and a precedence profile $\bar{\rhd}$). For a fixed school $a^* \in A$, we let $T_{a^*} \subseteq I$ be the set of students rejected by a^* during the cumulative offer process under choice functions $\bar{C}$. Formally, we have $T_{a^*} = \{i \in I : a^* P^i \bar{\mu}_i\}$. We fix some $\hat{T}_{a^*} \subseteq T_{a^*}$ and construct a **copy economy** with set of schools A, set of slots S, and precedence order profile $\bar{\rhd}$. The set of students in the copy economy, denoted $\check{I}$, is obtained by replacing each student $i \in \hat{T}_{a^*}$ with a **top copy** i^{t} and a **bottom copy** i^{b}:

$$\check{I} = (I \setminus \hat{T}_{a^*}) \bigcup \left(\cup_{i \in \hat{T}_{a^*}} \{i^{\mathsf{t}}, i^{\mathsf{b}}\} \right).$$

The relationship between the set $\check{I}$ of students in the copy economy and the set I of students in the original economy is pictured in Figure 3. Note that we suppress the dependence of $\check{I}$ on $\hat{T}_{a^*}$, as the set $\hat{T}_{a^*}$ under consideration is clearly identified whenever we undertake a copy economy construction. For a copy i^{c} of a student $i \in \hat{T}_{a^*}$ (where $\mathsf{c} \in \{\mathsf{t}, \mathsf{b}\}$), we say that i is the student **underlying** i^{c}.

Copies' preferences correspond to specific (post- and pre-)truncations of their underlying students' preferences, as pictured in Figure 4. The preference relation $P^{i^{\mathsf{t}}}$ of the top copy of i is defined so that

- $a'' P^{i^{\mathsf{t}}} a' \iff a'' P^i a'$ for all $a', a'' \in A$, and
- $a_0 P^{i^{\mathsf{t}}} a' \iff a^* P^i a'$ for all $a' \in A$.

[25] This observation follows from the facts—proven respectively in Proposition 3 and Lemma D.1 of Kominers and Sönmez (2013)—that the choice functions in our setting satisfy both the *substitutability* condition of Hatfield and Milgrom (2005) and the *irrelevance of rejected contracts* condition of Aygün and Sönmez (2013).

$$P^i : \underbrace{a^1 \succ \cdots \succ a^\ell \succ a^*}_{P^{i^{\mathsf{t}}}} \succ \underbrace{a^{\ell+1} \succ \cdots}_{P^{i^{\mathsf{b}}}}$$

Figure 4: Construction of copies' preference relations: $P^{i^{\mathsf{t}}}$ is the "top part" of P^i that ranks all the schools that i (weakly) prefers to a^*, while $P^{i^{\mathsf{b}}}$ is the "bottom part" of P^i that ranks all the schools that i likes (strictly) less than a^*. (Note that we must append a_0 just after a^* in $P^{i^{\mathsf{t}}}$.)

$$\cdots \bar{\pi}^s \underbrace{i}_{i^{\mathsf{t}}\check{\pi}^s i^{\mathsf{b}}} \bar{\pi}^s \bar{i} \bar{\pi}^s \cdots$$

Figure 5: Construction of priorities in the copy economy: $\check{\pi}^s$ is constructed so that each instance of a student $i \in \hat{T}_{a^*}$ in priority order $\bar{\pi}^s$ is replaced with the "subrelation" $i^{\mathsf{t}}\check{\pi}^s i^{\mathsf{b}}$.

(Roughly speaking, $P^{i^{\mathsf{t}}}$ is the "top part" of P^i that ranks all the schools that i (weakly) prefers to a^*.) The preference relation $P^{i^{\mathsf{b}}}$ of the bottom copy of i is defined similarly, with

- $a' R^i a^* \implies a_0 P^{i^{\mathsf{b}}} a'$ for all $a' \in A$, and
- $a^* P^i a' P^i a'' \implies a' P^{i^{\mathsf{b}}} a''$ for all $a', a'' \in (A \cup \{a_0\})$.

(Roughly speaking, $P^{i^{\mathsf{b}}}$ is the "bottom part" of P^i that ranks all the schools that i likes (strictly) less than a^*.)

The priorities $\check{\pi}^s$ in the copy economy (for each $s \in S$) are constructed by replacing students in $\hat{T}_{a^*}$ with their copies as follows:

- $i^{\mathsf{t}}\check{\pi}^s i^{\mathsf{b}}$ for all $i \in \hat{T}_{a^*}$;
- $i\bar{\pi}^s\bar{i} \implies i\check{\pi}^s\bar{i}$ for all $i, \bar{i} \in (I \setminus \hat{T}_{a^*})$;
- $i\bar{\pi}^s\bar{i} \implies i\check{\pi}^s\bar{i}^{\mathsf{t}}\check{\pi}^s\bar{i}^{\mathsf{b}}$ for all $i \in (I \setminus \hat{T}_{a^*})$ and $\bar{i} \in \hat{T}_{a^*}$;
- $\bar{i}\bar{\pi}^s i \implies \bar{i}^{\mathsf{t}}\check{\pi}^s\bar{i}^{\mathsf{b}}\check{\pi}^s i$ for all $i \in (I \setminus \hat{T}_{a^*})$ and $\bar{i} \in \hat{T}_{a^*}$; and
- $i\bar{\pi}^s\bar{i} \implies i^{\mathsf{t}}\check{\pi}^s i^{\mathsf{b}}\check{\pi}^s\bar{i}^{\mathsf{t}}\check{\pi}^s\bar{i}^{\mathsf{b}}$ for all $i, \bar{i} \in \hat{T}_{a^*}$.

This construction is illustrated in Figure 5. We write $\check{C}$ for the profile of choice functions induced by the priorities $\check{\pi}$ and precedence profile $\triangleright$.

We say that a set of students $\bar{I} \subseteq I$ is **equivalent** to a set of students $\check{\bar{I}} \subseteq \check{I}$ **up to copies** if $\bar{I}$ and $\check{\bar{I}}$ share the same set of students in $I \setminus \hat{T}_{a^*}$, and the set of students underlying the set of copy students in $\check{\bar{I}}$ exactly equals $\bar{I} \cap \hat{T}_{a^*}$. That is, $\bar{I} \subseteq I$ is **equivalent to** $\boldsymbol{\check{\bar{I}} \subseteq \check{I}}$ **up to copies** if

- $[\bar{I} \cap (I \setminus \hat{T}_{a^*})] = [\check{\bar{I}} \cap (I \setminus \hat{T}_{a^*})]$ and
- $[\bar{I} \cap \hat{T}_{a^*}] = \left\{ i \in \hat{T}_{a^*} \subseteq I : i^{\mathsf{c}} \in \check{\bar{I}} \text{ for some } \mathsf{c} \in \{\mathsf{t}, \mathsf{b}\} \right\}$;

in this case, we write $\bar{I} \overset{\text{cp}}{=} \check{\bar{I}}$.

Because the priorities $\check{\pi}^s$ are constructed so that each instance of a student $i \in \hat{T}_{a^*}$ in priority order $\bar{\pi}^s$ is replaced with the "subrelation" $i^{\mathsf{t}} \check{\pi}^s i^{\mathsf{b}}$, the following lemma is immediate.

Lemma 3. *Suppose that $\bar{I} \overset{\text{cp}}{=} \check{\bar{I}}$, and suppose moreover that for each $i \in \hat{T}_{a^*}$, there is at most one copy of i in $\check{\bar{I}}$. Then, for each school $a' \in A$, we have $\bar{C}^{a'}(\bar{I}) \overset{\text{cp}}{=} \check{C}^{a'}(\check{\bar{I}})$.*

We let $\check{\mu}$ be the outcome of the cumulative offer process in the copy economy under choice functions $\check{C}$. We now show that $\check{\mu}$ in a certain sense corresponds to $\bar{\mu}$ under copy equivalence.[26] Specifically, we show that under $\check{\mu}$ (in the copy economy):

- students in $I \setminus \hat{T}_{a^*}$ have the same assignments as under $\bar{\mu}$;
- all the top copies of students in $\hat{T}_{a^*}$ are assigned to a_0; and
- all the bottom copies of students in $\hat{T}_{a^*}$ receive the assignments that their underlying students receive under $\bar{\mu}$.

Lemma 4. *We have:*

- *$\check{\mu}_i = \bar{\mu}_i$ for any $i \in (I \setminus \hat{T}_{a^*})$;*
- *$\check{\mu}_{i^{\mathsf{t}}} = a_0$ for any $i \in \hat{T}_{a^*}$; and*
- *$\check{\mu}_{i^{\mathsf{b}}} = \bar{\mu}_i$ for any $i \in \hat{T}_{a^*}$.*

Consequently, for each $a' \in (A \cup \{a_0\})$, we have $\bar{\mu}_{a'} \overset{\text{cp}}{=} \check{\mu}_{a'}$.

Proof. We let $\bar{\Sigma} = \left\langle (i^1 \to a^1), (i^2 \to a^2), \ldots, (i^L \to a^L) \right\rangle$ be a (full) proposal sequence that can arise in the cumulative offer process under choice functions $\bar{C}$ (i.e., in the original economy).[27] Now, for each $\ell \leq L$ for which $i^\ell \in \hat{T}_{a^*}$, we let

$$\check{i}^\ell = \begin{cases} i^{\ell \mathsf{t}} & a^\ell R^{i^\ell} a^* \\ i^{\ell \mathsf{b}} & a^* P^{i^\ell} a^\ell. \end{cases}$$

For each ℓ for which $i^\ell \in (I \setminus \hat{T}_{a^*})$, we let $\check{i}^\ell = i^\ell$.

Claim. *The proposal sequence*

$$\check{\Sigma} = \left\langle (\check{i}^1 \to a^1), (\check{i}^2 \to a^2), \ldots, (\check{i}^\ell \to a^\ell), \ldots, (\check{i}^L \to a^L) \right\rangle \oplus \left\langle (i^{\mathsf{t}} \to a_0) : i \in \hat{T}_{a^*} \right\rangle \tag{11}$$

is a valid sequence of proposals for the cumulative offer process under choice functions $\check{C}$ (in the copy economy).[28]

[26] Recall that $\bar{\mu}$ is the outcome of cumulative offer process under choice functions $\bar{C}$.

[27] As the cumulative offer process outcome is independent of the proposal order in our context, we can analyze an arbitrary proposal sequence here.

[28] Here, $\oplus$ denotes the concatenation of sequences. Note that the ordering of the proposals in the appended subsequence $\left\langle (i^{\mathsf{t}} \to a_0) : i \in \hat{T}_{a^*} \right\rangle$ can be arbitrary.

Proof. For each ℓ and $a' \in (A \cup \{a_0\})$, we let $\bar{A}^{\ell}_{a'}$ denote the sets of available students arising in the cumulative offer process (in the original economy) under proposal order $\bar{\Sigma}$, and let $\check{A}^{\ell}_{a'}$ denote the sets of available students arising in the cumulative offer process (in the copy economy) under proposal order $\check{\Sigma}$.[29] We proceed by induction, showing that for each $\ell \leq L$,

1. $(\check{i}^{\ell} \to a^{\ell})$ is a valid proposal in step ℓ of the cumulative offer process (in the copy economy), and

2. for each $a' \in (A \cup \{a_0\})$, we have
$$\bar{A}^{\ell+1}_{a'} \overset{\text{cp}}{=} \check{A}^{\ell+1}_{a'}. \tag{12}$$

Both hypotheses are clearly true in the base case $\ell = 1$, so we assume that they hold up to ℓ, and show that this implies them in the case $\ell + 1$.

First, we note that for each $i \in I$ and $a' \in A$, the proposal $(i \to a')$ occurs in the sequence $\bar{\Sigma}$ at most once, as no student ever proposes to the same school twice in the cumulative offer process in the original economy. Thus, by our construction of $\check{\Sigma}$, we see that there is no student $i \in \hat{T}_{a^*}$ for whom two distinct copies propose to any school $a' \in A$. It follows that for each $i \in \hat{T}_{a^*}$ and $a' \in A$, there is at most one copy of i in $\check{A}^{\ell+1}_{a'}$. Thus, the conclusion of Lemma 3 applies: For each $a' \in A$, we have
$$\bar{C}^{a'}(\bar{A}^{\ell+1}_{a'}) \overset{\text{cp}}{=} \check{C}^{a'}(\check{A}^{\ell+1}_{a'}). \tag{13}$$

Now, if $(i^{\ell+1} \to a^{\ell+1})$ is a valid proposal at step $\ell + 1$ of the cumulative offer process in the original economy, then $i^{\ell+1}$ is not held by any school $a' \in (A \cup \{a_0\})$ at the end of step ℓ of the cumulative offer process in the original economy, i.e.,
$$i^{\ell+1} \notin \left[\left(\bigcup_{a' \in A} \bar{C}^{a'}(\bar{A}^{\ell+1}_{a'}) \right) \cup \bar{A}^{\ell+1}_{a_0} \right]; \tag{14}$$
moreover, $i^{\ell+1}$ has not proposed to school $a^{\ell+1}$ by the end of step ℓ of that process.

If $i^{\ell+1} \in (I \setminus \hat{T}_{a^*})$, then we see immediately from (13), (14), and (12) (in the case $a' = a_0$) that
$$(\check{i}^{\ell+1} \to a^{\ell+1}) = (i^{\ell+1} \to a^{\ell+1})$$
is a valid proposal at step $\ell + 1$ of the cumulative offer process in the copy economy. Moreover, it follows from (12) (and the fact that $i^{\ell+1} = \check{i}^{\ell+1}$) that
$$\bar{A}^{\ell+2}_{a^{\ell+1}} = (\bar{A}^{\ell+1}_{a^{\ell+1}} \cup \{i^{\ell+1}\}) \overset{\text{cp}}{=} (\check{A}^{\ell+1}_{a^{\ell+1}} \cup \{i^{\ell+1}\}) = (\check{A}^{\ell+1}_{a^{\ell+1}} \cup \{\check{i}^{\ell+1}\}) = \check{A}^{\ell+2}_{a^{\ell+1}}.$$
As for each $a' \neq a^{\ell+1}$ we have
$$\bar{A}^{\ell+2}_{a'} = \bar{A}^{\ell+1}_{a'} \overset{\text{cp}}{=} \check{A}^{\ell+1}_{a'} = \check{A}^{\ell+2}_{a'}$$
(using (12) for the middle equality), we find that
$$\bar{A}^{\ell+2}_{a'} \overset{\text{cp}}{=} \check{A}^{\ell+2}_{a'}$$

[29] Formally, the sets $\bar{A}^{\ell}_{a'}$ are given, and we construct the sets $\check{A}^{\ell}_{a'}$ inductively, as we show by induction that $\check{\Sigma}$ is a valid cumulative offer process proposal order.

for each $a' \in (A \cup \{a_0\})$, as desired.

Now, we suppose instead that $i^{\ell+1} \in \hat{T}_{a^*}$. Then from (13), (14), and (12) (in the case $a' = a_0$), we see that no copy of $i^{\ell+1}$ is held by any school $a' \in (A \cup \{a_0\})$ at the end of step ℓ of the cumulative offer process in the copy economy.[30] Noting that no top copy i^{t} proposes to a_0 until after proposal $(\check{i}^L \to a^L)$, we see that both the top and bottom copies of $i^{\ell+1}$ are available to propose at the beginning of step $\ell+1$ of the cumulative offer process in the copy economy; hence, $\check{i}^{\ell+1}$ is available to propose at step $\ell+1$, irrespective of which copy of $i^{\ell+1}$ he or she is. Combining the preceding observations, we see that $(\check{i}^{\ell+1} \to a^{\ell+1})$ is a valid proposal at step $\ell+1$ of the cumulative offer process in the copy economy. We then have from (12) (and the fact that $\{i^{\ell+1}\} \overset{\text{cp}}{=} \{\check{i}^{\ell+1}\}$) that

$$\bar{A}^{\ell+2}_{a^{\ell+1}} = (\bar{A}^{\ell+1}_{a^{\ell+1}} \cup \{i^{\ell+1}\}) \overset{\text{cp}}{=} (\check{A}^{\ell+1}_{a^{\ell+1}} \cup \{i^{\ell+1}\}) \overset{\text{cp}}{=} (\check{A}^{\ell+1}_{a^{\ell+1}} \cup \{\check{i}^{\ell+1}\}) = \check{A}^{\ell+2}_{a^{\ell+1}},$$

and so we find that

$$\bar{A}^{\ell+2}_{a'} \overset{\text{cp}}{=} \check{A}^{\ell+2}_{a'}$$

for each $a' \in (A \cup \{a_0\})$, as desired.

The preceding observations show that

$$\Big\langle (\check{i}^1 \to a^1), (\check{i}^2 \to a^2), \ldots, (\check{i}^\ell \to a^\ell), \ldots, (\check{i}^L \to a^L) \Big\rangle \tag{15}$$

is a valid sequence of proposals for the cumulative offer process in the copy economy. Now, we note that following those proposals, all students in $(I \setminus \hat{T}_{a^*}) \cup \{i^{\mathsf{b}} : i \in \hat{T}_{a^*}\}$ are held by (possibly null) schools, and all students in $\{i^{\mathsf{t}} : i \in \hat{T}_{a^*}\}$ are available to propose again.

The final proposal of each top copy $\bar{i}^{\mathsf{t}} \in \{i^{\mathsf{t}} : i \in \hat{T}_{a^*}\}$ prior to step L is $(\bar{i}^{\mathsf{t}} \to a^*)$, by construction of (15). As only the top copies $\bar{i}^{\mathsf{t}} \in \{i^{\mathsf{t}} : i \in \hat{T}_{a^*}\}$ are available to propose in step $L+1$ and a_0 is ranked just after a^* in all top copies' preference relations, the cumulative offer process in the copy economy is completed by running the sequence of proposals $\Big\langle (i^{\mathsf{t}} \to a_0) : i \in \hat{T}_{a^*} \Big\rangle$ (in any order). $\square$

Now, we observe that under the proposal sequence $\check{\Sigma}$ defined by (11):

O1. The last school each student $i \in (I \setminus \hat{T}_{a^*})$ proposes to is the (possibly null) school that i proposes to last in the cumulative offer process in the original economy (under proposal sequence $\bar{\Sigma}$).

O2. For each $i \in \hat{T}_{a^*}$, the last school i^{t} proposes to is a_0.

O3. For each $i \in \hat{T}_{a^*}$, the last school i^{b} proposes to is the (possibly null) school that i proposes to last in the cumulative offer process in the original economy (under proposal sequence $\bar{\Sigma}$).

Now, *any* valid cumulative offer process proposal sequence in the copy economy yields the outcome $\check{\mu}$. Thus, in particular, we see that $\check{\mu}$ is the outcome of the cumulative offer process in the copy economy under proposal sequence $\check{\Sigma}$. The desired result then follows from observations O1–O3. $\square$

[30] Here, we implicitly also use the hypothesis of the claim that all proposals of top copies to the null school are executed at the end of the proposal sequence.

B.3.3 Main Argument

In the sequel, we assume the setup of either Section B.1 or Section B.2, let $C^{a'} = D^{a'}$ for all (nonnull) schools $a' \neq a$, and let μ and ν respectively denote the cumulative offer process outcomes under the choice functions C and D. We make use of an **Adjustment Lemma**, which is Lemma 1 for the case of Proposition 3.(i) and Lemma 2 for the case of Proposition 3.(ii). For each $a' \in A$ and any matching $\bar{\mu}$, we denote by

$$\mathfrak{n}_{a'}(\bar{\mu}) \equiv |\bar{\mu}_{a'} \cap I_{a'}|$$

the number of students that are reserve-eligible at a' and are assigned to a' under matching $\bar{\mu}$.

First, we note the following immediate corollary of the Adjustment Lemma.

Lemma 5. *For $\bar{I} \subseteq I$, if $|\bar{I}| > q_a$, then $|[\bar{I} \setminus (C^a(\bar{I}))] \cap [\bar{I} \setminus (D^a(\bar{I}))]| \geq |\bar{I}| - q_a - 1$.*[31]

Now, we let $\hat{T}_a \subseteq I$ be the set of students who are rejected from a in *both* the cumulative offer process under choice functions C *and* the cumulative offer process under choice functions D. As the students in $\hat{T}_a$ are rejected in the cumulative offer process under *both* choice functions C and D, we may consider the copy economy associated with the original economy by the construction introduced in Section B.3.2, for both choice function profiles. We denote by $\check{I}$ the set of students in the copy economy, and denote by $\check{C}$ and $\check{D}$ the copy economy choice functions associated to C and D, respectively.

We denote by $\check{\mu}$ and $\check{\nu}$ the (copy economy) cumulative offer process outcomes under choice functions $\check{C}$ and $\check{D}$, respectively. By Lemma 4, we have

- $\check{\mu}_i = \mu_i$ and $\check{\nu}_i = \nu_i$ for any $i \in (I \setminus \hat{T}_a)$;
- $\check{\mu}_{i^{\mathsf{t}}} = a_0$ and $\check{\nu}_{i^{\mathsf{t}}} = a_0$ for any $i \in \hat{T}_a$; and
- $\check{\mu}_{i^{\mathsf{b}}} = \mu_i$ and $\check{\nu}_{i^{\mathsf{b}}} = \nu_i$ for any $i \in \hat{T}_a$.

As every student in $\hat{T}_a$ is rejected from a in the cumulative offer process under choice functions C, we have

$$[\mu_a \cap I_a] = [(\mu_a \cap I_a) \setminus \hat{T}_a] = [\mu_a \cap (I_a \setminus \hat{T}_a)] = [\check{\mu}_a \cap (I_a \setminus \hat{T}_a)] = [\check{\mu}_a \cap I_a], \tag{16}$$

where the second-to-last equality follows from the fact that $\check{\mu}_i = \mu_i$ for each $i \in (I_a \setminus \hat{T}_a)$, and the last equality follows because $[\check{\mu}_a \cap \hat{T}_a] = \emptyset$. It therefore follows that

$$\mathfrak{n}_a(\mu) = |\mu_a \cap I_a| = |\check{\mu}_a \cap I_a|. \tag{17}$$

Analogously, we find that

$$\mathfrak{n}_a(\nu) = |\nu_a \cap I_a| = |\check{\nu}_a \cap I_a|. \tag{18}$$

Thus, to show our proposition it suffices to prove that weakly more students that are reserve-eligible at a are assigned to a under $\check{\nu}$ than under $\check{\mu}$. To show this, we recall that cumulative offer processes

[31] Formally, this is also true in the case that $|\bar{I}| \leq q_a$, as then $|\bar{I}| - q_a - 1 < 0$.

are always independent of proposal order and consider particular orders for the (copy economy) cumulative offer processes under choice functions $\check{C}$ and $\check{D}$.

Under each process, we first execute as many proposals ($\check{i} \to a'$) as possible with $\check{i} \in \check{I}$ and $a' P^{\check{i}} a$; since $\check{C}$ and $\check{D}$ differ only with respect to $\check{C}^a$ and $\check{D}^a$, in each of the cumulative offer processes, we can use the exact same order of proposals for this initial sequence. Once such proposals are completed, each student in $\check{I}$ either

- is on hold at some (possibly null) school $a' \neq a$, or
- has proposed to all schools he/she prefers to a, and is available to propose to a;

we let $\check{J}$ be the set of students in the latter of these two categories. By construction, at this stage, the sets of students on hold at schools $a' \neq a$ (including a_0) are the same in both processes.[32] Also, $\check{J}$ contains no bottom copies of any student $i \in \hat{T}_a$, as bottom copies do not find a acceptable.

We continue the cumulative offer processes by having the students in $\check{J}$ propose to a in uninterrupted sequence. Following these proposals, the set of students available to a is exactly $\check{J}$.

If $|\check{J}| \leq q_a$, then all students in $\check{J}$ are held by a, and both processes terminate after all the students in $\check{J}$ have proposed to a. In this case, we have $\check{\mu}_a = \check{\nu}_a$ (indeed, $\check{\mu} = \check{\nu}$); hence (17) and (18) together show that $\mathfrak{n}_a(\mu) = \mathfrak{n}_a(\nu)$.

If instead we have $|\check{J}| > q_a$, then we examine the set

$$\check{K} \equiv [(\check{J} \setminus \check{C}^a(\check{J})) \cap (\check{J} \setminus \check{D}^a(\check{J}))] \tag{19}$$

of students rejected under both $\check{C}^a$ and $\check{D}^a$ when (exactly) the set of students $\check{J}$ is available. By construction, $\check{K}$ is copy-equivalent to a subset of $\hat{T}_a$. Thus, we see that $\check{K}$ must consist entirely of top copies of students in $\hat{T}_a$, as represented in the exterior box of Figure 6. All such copies rank a_0 immediately below a. Hence, we may continue both cumulative offer processes by having all of these students propose to a_0; we execute all such proposals.

Recall that up to this point, we have executed the cumulative offer processes under choice functions $\check{C}$ and $\check{D}$ in complete, step-by-step parallel. The sets of students held by each school $a' \neq a$ (including a_0) are exactly the same; meanwhile, a holds $\check{C}^a(\check{J})$ in the process under choice functions $\check{C}$ and holds $\check{D}^a(\check{J})$ in the process under choice functions $\check{D}$.

Combining Lemma 5 with the fact that a fills all its slots when possible, we can bound the size of $\check{K}$ from below and above: we have

$$|\check{J}| - q_a - 1 \leq |\check{K}| \leq |\check{J}| - q_a, \tag{20}$$

as pictured in the interior of Figure 6. From (20), we see that we have two cases to consider.

[32] We return to this stage in our proof of Proposition 4.

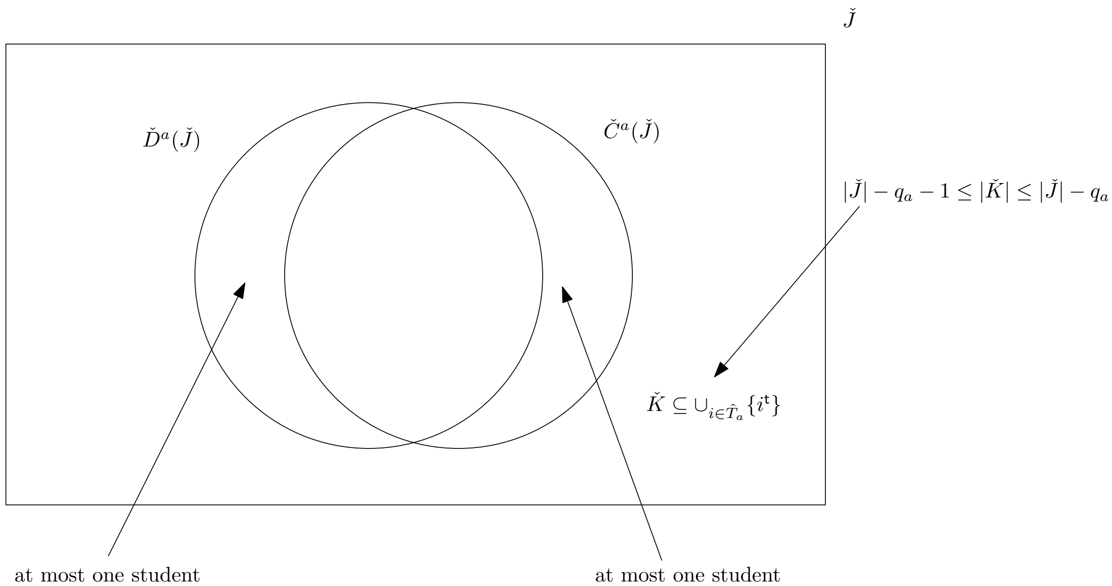


Figure 6: The structure of the choices of a from $\check{J}$ under choice functions $\check{D}^a$ and $\check{C}^a$, as implied by the Adjustment Lemma.

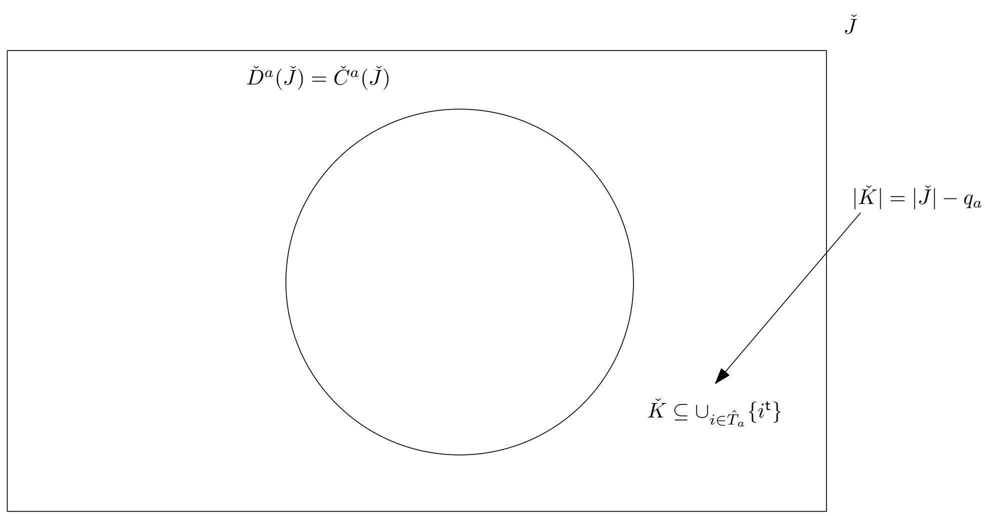


Figure 7: Case 1: $|\check{K}| = |\check{J}| - q_a$ and $\check{C}^a(\check{J}) = \check{D}^a(\check{J})$.

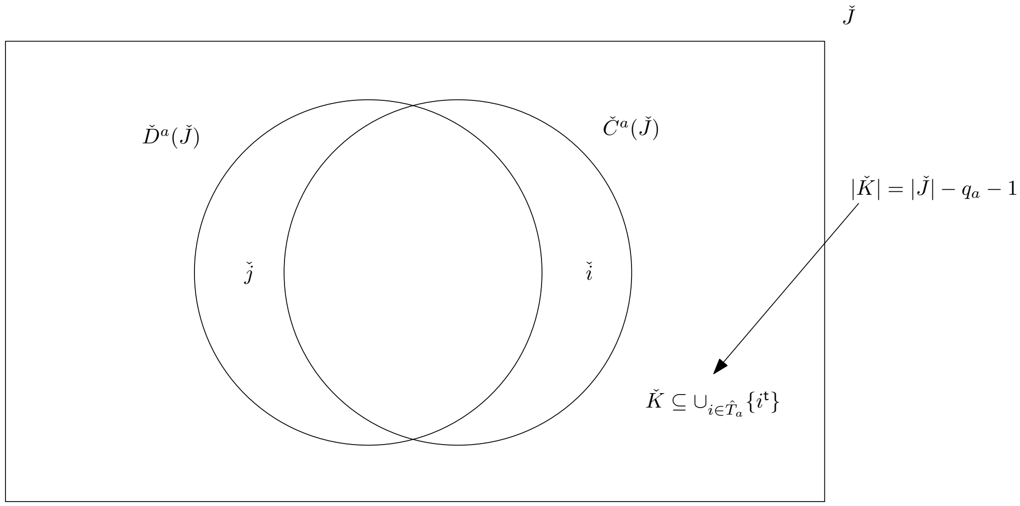


Figure 8: Case 2: $|\check{K}| = |\check{J}| - q_a - 1$ and $\check{C}^a(\check{J}) \neq \check{D}^a(\check{J})$.

Case 1: $|\check{K}| = |\check{J}| - q_a$**.** If $|\check{K}| = |\check{J}| - q_a$, then (again because a fills all its slots when possible) we must have $\check{C}^a(\check{J}) = \check{D}^a(\check{J})$, as pictured in Figure 7. In this case, the cumulative offer processes under $\check{C}$ and $\check{D}$ terminate after the final proposals of students in $\check{K}$ (which can be processed in the same order under choice functions $\check{C}$ and $\check{D}$); we then have $\check{\mu}_a = \check{\nu}_a$ (indeed, $\check{\mu} = \check{\nu}$), which again shows that $\mathfrak{n}_a(\mu) = \mathfrak{n}_a(\nu)$.

Case 2: $|\check{K}| = |\check{J}| - q_a - 1$**.** If, instead $|\check{K}| = |\check{J}| - q_a - 1$, then $\check{C}^a(\check{J}) \neq \check{D}^a(\check{J})$. By the Adjustment Lemma, we see that

$$|(\check{D}^a(\check{J})) \cap I_a| > |(\check{C}^a(\check{J})) \cap I_a|. \tag{21}$$

Moreover, the Adjustment Lemma shows that there is a unique student

$$\check{i} \in [[(\check{C}^a(\check{J})) \cap (I \setminus I_a)] \setminus [(\check{D}^a(\check{J})) \cap (I \setminus I_a)]]$$

and a unique student

$$\check{j} \in [[(\check{D}^a(\check{J})) \cap I_a] \setminus [(\check{C}^a(\check{J})) \cap I_a]],$$

as pictured in Figure 8.

- First, we suppose that $\check{J} \setminus \check{K}$ contains no copies of students in $\hat{T}_a$. Then, in particular, neither $\check{i}$ nor $\check{j}$ is a copy of a student in $\hat{T}_a$.

  As $\check{i}$ is rejected by a in the cumulative offer process under choice functions $\check{D}$, we know that $aP^{\check{i}}\check{\nu}_{\check{i}} = \nu_{\check{i}}$; it follows that $\check{i}$ is also rejected in the cumulative offer process under choice functions D. But $\check{i} \notin \hat{T}_a$, so we know that $\check{i}$ is *not* rejected in the cumulative offer process under choice functions C. Thus, $\mu_{\check{i}} R^{\check{i}} a$. As $\check{\mu}_{\check{i}} = \mu_{\check{i}}$ and $\check{i}$ proposes to a in the cumulative offer process under choice functions $\check{C}$, we find that we must have $\check{\mu}_{\check{i}} = a$, that is, $\check{i}$ is not rejected by a in the remainder of the cumulative offer process under choice functions $\check{C}$. By our choice of $\check{i}$, however, we know that $\check{i} \in (I \setminus I_a)$ and that $\check{i}$ has the lowest rank under π^o among all reserve-ineligible students in $\check{C}^a(\check{J})$. It follows that the number of reserve-ineligible students assigned to a weakly increases throughout the remainder of the cumulative offer process under choice functions $\check{C}$, that is,

$$|\check{\mu}_a \cap (I \setminus I_a)| \geq |(\check{C}^a(\check{J})) \cap (I \setminus I_a)|.$$

This implies that

$$|\check{\mu}_a \cap I_a| \leq |(\check{C}^a(\check{J})) \cap I_a|. \tag{22}$$

Analogously, we find that $\check{\nu}_{\check{j}} = a$, which implies that

$$|\check{\nu}_a \cap I_a| \geq |(\check{D}^a(\check{J})) \cap I_a|. \tag{23}$$

Now, we find that

$$\begin{aligned} \mathfrak{n}_a(\nu) &= |\check{\nu}_a \cap I_a| &(24)\\ &\geq |(\check{D}^a(\check{J})) \cap I_a| &(25)\\ &> |(\check{C}^a(\check{J})) \cap I_a| &(26)\\ &\geq |\check{\mu}_a \cap I_a| &(27)\\ &= \mathfrak{n}_a(\mu), &(28) \end{aligned}$$

as desired, where (24) follows from (18), (25) follows from (23), (26) follows from (21), (27) follows from (22), and (28) follows from (17).

- Finally, we consider the case in which $\check{J} \setminus \check{K}$ contains at least one copy of a student in $\hat{T}_a$. By construction, any such copy must be a top copy.

  Claim. *If some copy $i^{\mathsf{t}} \notin \{\check{i}, \check{j}\}$ is in $\check{J} \setminus \check{K}$ for some $i \in \hat{T}_a$, then at least one of $\check{i}$ or $\check{j}$ must be a (top) copy, as well.*

  Proof. We suppose first that the student i underlying i^{t} is reserve-eligible at a, i.e., $i \in I_a$, but that $\check{j}$ is not a copy. Then, since $i^{\mathsf{t}} \in [(\check{J} \setminus \check{K}) \setminus \{\check{i}, \check{j}\}]$, we know that $i^{\mathsf{t}} \in (\check{C}^a(\check{J}))$; in particular, i has higher rank under π^o than $\check{j}$. As $i \in \hat{T}_a$, we know that i is rejected in the cumulative offer processes under both choice functions C and D. It follows that the lower-π^o-ranked student $\check{j}$ must also be rejected in the cumulative offer processes under both choice functions C and D, as he/she proposes to a in each of those processes; hence, we must have $\check{j} \in \hat{T}_a$—but this is impossible, as otherwise $\check{j}$ would have to be a copy (as all students in $\hat{T}_a$ are represented by copies in our chosen copy economy). An analogous argument shows that if the student i underlying i^{t} is reserve-ineligible at a, then $\check{i}$ must be a (top) copy. □

  Claim. *There is at most one (top) copy in $\check{J} \setminus \check{K}$.*

Proof. We suppose there are at least two (top) copies in $\check{J} \setminus \check{K}$. By the preceding claim, either $\check{i}$ or $\check{j}$ must be a (top) copy. We assume the former case ($\check{i}$ is a copy); the argument in the latter case is analogous. We let i^{t} be a (top) copy in $\check{J} \setminus \check{K}$ with $i^{\mathsf{t}} \neq \check{i}$.

As $\check{i} \notin \check{D}^a(\check{J})$, we may continue the cumulative offer process under choice functions $\check{D}$ (after having all students in $\check{K}$ propose to a_0) by having $\check{i}$ apply to his or her next-most-preferred school after a—since $\check{i}$ is a top copy, this school is a_0. At this point, $\check{i}$ is held by a_0. At the end of this cumulative offer process step, a must hold all the students in $(\check{J} \setminus \check{K}) \setminus \{\check{i}\}$, or else a holds fewer than q_a students. But this means that the process terminates, as all students are held by schools. We therefore have $\check{\nu}_{i^{\mathsf{t}}} = a$. This contradicts the fact that we must have $\check{\nu}_{i^{\mathsf{t}}} = a_0$ (by Lemma 4), as i^{t} is a top copy. □

The preceding claims show that there is exactly one (top) copy in $\check{J} \setminus \check{K}$, and that it is either $\check{i}$ or $\check{j}$. We assume the former case ($\check{i}$ is a copy); the argument in the latter case is analogous. Now, we may continue the cumulative offer process under choice functions $\check{D}$ (after having all students in $\check{K}$ propose to a_0) by having $\check{i}$ propose to his/her next-most-preferred school after a—as $\check{i}$ is a top copy, this school is a_0, and the process terminates after the $(\check{i} \to a_0)$ proposal. Then, we have $\check{\nu}_a = \check{D}^a(\check{J})$, so

$$|\check{\nu}_a \cap (I \setminus I_a)| = |(\check{D}^a(\check{J})) \cap (I \setminus I_a)| = |(\check{C}^a(\check{J})) \cap (I \setminus I_a)| - 1. \tag{29}$$

Meanwhile, the student underlying $\check{i}$ has lower rank under π^o than any reserve-ineligible student in $\check{J} \setminus \check{K}$. It follows that $\check{i}$ will be the *first* student in $(\check{J} \setminus \check{K}) \setminus I_a$ rejected from a in the remainder of the cumulative offer process under choice functions $\check{C}$. After such a rejection occurs, we may have $\check{i}$ propose to his or her next-most-preferred school after a—as above, as $\check{i}$ is a top copy, this school is a_0 and the process terminates after the $(\check{i} \to a_0)$ proposal. Thus, we see that

$$|\check{\mu}_a \cap (I \setminus I_a)| \geq |[(\check{C}^a(\check{J})) \setminus \{\check{i}\}] \cap (I \setminus I_a)| = |(\check{C}^a(\check{J})) \cap (I \setminus I_a)| - 1. \tag{30}$$

Combining (29) and (30), we see that

$$|\check{\nu}_a \cap (I \setminus I_a)| = |(\check{C}^a(\check{J})) \cap (I \setminus I_a)| - 1 \leq |\check{\mu}_a \cap (I \setminus I_a)|;$$

it follows that

$$|\check{\nu}_a \cap I_a| \geq |\check{\mu}_a \cap I_a|. \tag{31}$$

Combining (31) with (17) and (18), we find that $\mathfrak{n}_a(\nu) \geq \mathfrak{n}_a(\mu)$; this completes our argument.

B.4 Proof of Proposition 4

We prove Proposition 4 by further analyzing the proof of Proposition 3—particularly, the end of Section B.3.3—in the case of uniform reserve priority. As such, we maintain the notations and conventions of Section B.3 and add the additional hypothesis that

$$\text{for any two schools } a', a'' \in A, \text{ we have } I_{a'} = I_{a''}. \tag{32}$$

By (32), the number of reserve-eligible students matched under any matching $\bar{\mu}$ is exactly

$$\left|\bigcup_{a' \in A} (\bar{\mu}_{a'} \cap I_{a'})\right| = \left|\bigcup_{a' \in A} (\bar{\mu}_{a'} \cap I_a)\right|;$$

hence, to prove the desired result we just need to show that at most one more student in I_a is matched in μ (that is, under C) than in ν (that is, under D).

We suppose that we have run the cumulative offer processes as in the proof of Proposition 3 up to the last point at which the cumulative offer processes under choice functions $\check{C}$ and $\check{D}$ can be run in complete, step-by-step parallel. Thus, as at the point marked by Footnote 32, the sets of students held by each school $a' \neq a$ (including a_0) are exactly the same under the two cumulative offer processes; meanwhile, a holds $\check{C}^a(\check{J})$ in the process under choice functions $\check{C}$ and holds $\check{D}^a(\check{J})$ in the process under choice functions $\check{D}$.

We recall the construction of $\check{J}$ (presented just before Footnote 32) and the subsequent construction of $\check{K}$ (presented in (19)). If $|\check{J}| \leq q_a$ or $|\check{K}| = |\check{J}| - q_a$, then $\check{\mu} = \check{\nu}$, so we have $\mu = \nu$ by Lemma 4, and the result is immediate.

Now, we consider the case in which $|\check{K}| = |\check{J}| - q_a - 1$. As all the students in $\check{K}$ are top copies, we know that they are assigned to a_0 in the cumulative offer processes under both $\check{C}$ and $\check{D}$. Thus, when we resume our analysis, all students except our $\check{i}$ and $\check{j}$ are on hold under both processes—so the cumulative offer process under $\check{C}$ will terminate whenever some student[33] applies to either a_0 or a school that is not filled to capacity, and the cumulative offer process under $\check{D}$ will, as well.

If the cumulative offer process under $\check{D}$ terminates with some student proposing to a school that is not filled to capacity, then all students matched under $\check{\mu}$ are matched under $\check{\nu}$; in this case, as the same students are reserve-eligible at all schools (by our uniform reserve priority assumption), we know that weakly more reserve-eligible students must be matched under $\check{\nu}$ than under $\check{\mu}$. By Lemma 4, then, the number of reserve-eligible students matched in ν must be at least as large as the number of reserve-eligible students matched in μ.

If the cumulative offer process under $\check{D}$ terminates with a reserve-ineligible student proposing to a_0, then all reserve-eligible students matched under $\check{\mu}$ are matched under $\check{\nu}$, and—again by Lemma 4—weakly more reserve-eligible students are matched in ν than under μ.

Finally, if the cumulative offer process under $\check{D}$ terminates with a reserve-eligible student proposing to a_0, we have two cases to consider: First, if the cumulative offer process under $\check{C}$ terminates with a reserve-ineligible student proposing to a_0 or to a school that is not filled to capacity, then there is at most one more reserve-eligible student matched under $\check{\mu}$ than under $\check{\nu}$ (and hence, there is at most one more reserve-eligible student matched under μ than under ν). Second, if the cumulative offer process under $\check{C}$ terminates with a reserve-eligible student proposing to either a_0 or a school that is not filled to capacity, then again there is at most one more reserve-eligible student matched under $\check{\mu}$ than under $\check{\nu}$ (and hence, there is at most one more reserve-eligible student matched under μ than under ν).

[33] This is not necessarily $\check{j}$, but rather the last student in the rejection chain that $\check{j}$ initiates.

B.5 Proof of Proposition 5

We assume that there are only two (nonnull) schools, $A = \{a, b\}$, and moreover assume the setup of Section B.3: either an open slot of a is replaced with a reserve slot, or the precedence order position of a reserve slot of a is switched with that of a subsequent open slot; D and C are the associated choice functions (with $C^a \neq D^a$ and $C^b = D^b$); and ν and μ are the associated deferred acceptance/cumulative offer process outcomes.[34]

Lemma 6. *We have $|\nu_a| = |\mu_a|$ and $|\nu_b| = |\mu_b|$. That is, the number of slots filled at each school under μ is the same as under ν.*

Proof. First, we recall that the deferred acceptance/cumulative offer process outcomes in our context are **stable**, in the sense that they

- never assign students to schools that those students find unacceptable,
- eliminate justified envy among students, and
- never leave a demanded seat unfilled

(see Kominers and Sönmez (2013, 2016)).[35]

If *both* of the schools a and b have an empty slot under either matching ν or μ, then stability implies that all students get their first choices under each matching; hence $\nu = \mu$ and the result holds immediately. Likewise, if neither school has an empty slot under either matching, the result holds immediately, as then $|\nu_a| = |\mu_a| = |S^a|$ and $|\nu_b| = |\mu_b| = |S^b|$. Hence the only non-trivial case is when one school is full under one of the matchings but the other is not.

We suppose that under matching μ, school a has an empty slot, while school b fills all its slots; the argument in the case that school a fills all its slots under μ, while school b does not is analogous.

If school a has an empty slot under μ and b fills all its slots, then not only does each student who is assigned a slot at school b under matching μ prefer school b to school a, but also there are at least as many students whose first choice is school b as as there are slots at school b. Thus, stability implies that school b must fill all its slots under matching ν as well; hence, $|\nu_b| = |\mu_b| = |S^b|$. By assumption,

- there are at least as many slots as students, and
- all students find both schools acceptable.

[34] Note that choosing to adjust the choice function of school a (rather than that of b) is without loss of generality.

[35] In our context, the first condition is actually vacuous because we have assumed that all students find both schools a and b acceptable; we state the condition for completeness. The second and third conditions together mean that if

- i is assigned to school $c \in (A \cup \{a_0\})$ under the deferred acceptance/cumulative offer process outcome and
- prefers school $d \in A$ to c,

then all of d's slots must be filled, and all the students assigned to d must have higher priority for their slots than i does.

Therefore, we see that

$$|\nu_a| = |I| - |\nu_b| = |I| - |\mu_b| = |\mu_a|;$$

this observation completes the proof. □

B.5.1 Proof of Proposition 5.(i) and Proposition 5.(ii)

We prove Propositions 5.(i) and 5.(ii) using a completely parallel argument for the two results. We make use of an **Adjustment Proposition**, which is Proposition 3.(i) for the case of Proposition 5.(i), and Proposition 3.(ii) for the case of Proposition 5.(ii).

Proposition 6. *Weakly more students are assigned to schools at which they are reserve-eligible under ν than under μ; that is,*

$$\mathfrak{n}_a(\nu) + \mathfrak{n}_b(\nu) \geq \mathfrak{n}_a(\mu) + \mathfrak{n}_b(\mu).$$

Proof. If $\nu_a = \mu_a$, then we have

$$\nu_b = [I \setminus \nu_a] = [I \setminus \mu_a] = \mu_b,$$

as by assumption

- there are at least as many slots as students, and
- all students find both schools acceptable;

hence, the desired result is immediate.

If $\nu_a \neq \mu_a$, then

$$|\nu_a \cap I_a| = \mathfrak{n}_a(\nu) \geq \mathfrak{n}_a(\mu) = |\mu_a \cap I_a|$$

by the Adjustment Proposition. Thus, Lemma 6 implies that

$$|\nu_a \cap (I \setminus I_a)| = |\nu_a| - |\nu_a \cap I_a| \leq |\mu_a| - |\mu_a \cap I_a| = |\mu_a \cap (I \setminus I_a)|,$$

which in turn implies that

$$|\nu_a \cap I_b| \leq |\mu_a \cap I_b|$$

as $I \setminus I_a = I_b$ by assumption. Thus, we see that

$$\mathfrak{n}_b(\nu) = |\nu_b \cap I_b| = |I_b| - |\nu_a \cap I_b| \geq |I_b| - |\mu_a \cap I_b| = |\mu_b \cap I_b| = \mathfrak{n}_b(\mu)$$

as all students (and in particular all students in I_b) are matched under both μ and ν. Hence, we have

$$\mathfrak{n}_a(\nu) + \mathfrak{n}_b(\nu) \geq \mathfrak{n}_a(\mu) + \mathfrak{n}_b(\mu);$$

this completes the proof. □

B.5.2 Proof of Proposition 5.(iii)

Let r_a^1 denote the number of students who rank school a as first choice, and let r_b^1 denote the number of students who rank school b as first choice.

We work directly with the student-proposing deferred acceptance algorithm in this argument (rather than the cumulative offer process).

By assumption, $|S^a| + |S^b| \geq |I|$. Thus, as each student has exactly one first choice,

$$|S^a| + |S^b| \geq |I| = r_a^1 + r_b^1.$$

Hence, either:

1. $|S^a| \geq r_a^1$ and $|S^b| \geq r_b^1$, or

2. $|S^a| > r_a^1$ and $|S^b| < r_b^1$, or

3. $|S^a| < r_a^1$ and $|S^b| > r_b^1$.

In the first case, the student-proposing deferred acceptance algorithm terminates in one step and all students receive their first choices under both μ and ν. Thus, the result is immediate.

The analyses of the second and third cases are analogous, so it suffices to consider the case that $|S^a| > r_a^1$ and $|S^b| < r_b^1$.

Claim. *If $|S^a| > r_a^1$ and $|S^b| < r_b^1$, then under both μ and ν,*

- *the number of students that receive their first choices is equal to $|S^b| + r_a^1$, and*

- *the number of students that receive their second choices is equal to $r_b^1 - |S^b|$.*

Proof. We consider the construction of either μ or ν through the student-proposing deferred acceptance algorithm and observe that school b receives $r_b^1 > |S^b|$ offers in Step 1, holding $|S^b|$ of these while rejecting $r_b^1 - |S^b|$. School a, meanwhile, receives $r_a^1 < |S^a|$ offers and holds all of them. In Step 2, all students rejected by school b apply to school a, bringing the total number of applicants at school a to $r_a^1 + (r_b^1 - |S^b|)$. As $r_a^1 + (r_b^1 - |S^b|) \leq |S^a|$ by assumption, no student is rejected by school a, and the algorithm terminates in Step 2. Hence, under both μ and ν,

- $|S^b|$ students are assigned to school b as their first choice,

- r_a^1 students are assigned to school a as their first choice, and

- $r_b^1 - |S^b|$ students are assigned to school a as their second choice;

these observations show the claim. □

The preceding claim shows the desired result for the case that $|S^a| > r_a^1$ and $|S^b| < r_b^1$; an analogous argument shows the result for the case that $|S^a| < r_a^1$ and $|S^b| > r_b^1$, completing the proof.

C Official BPS 50-50 Policy

The official document describing the 50-50 policy states (BPS 1999):

> "Fifty percent walk zone preference means that half of the seats at a given school are subject to walk zone preference. The remaining seats are open to students outside of the walk zone.
>
> RATIONALE: One hundred percent walk zone preference in a controlled choice plan without racial guidelines could result in all available seats being assigned to students within the walk zone. The result would limit choice and access for all students, including those who have no walk zone school or live in walk zones where there are insufficient seats to serve the students residing in the walk zone.
>
> Patterns of parent choice clearly establish that many choose schools outside of their walk zone for many educational and other reasons. [...] One hundred percent walk zone preference would limit choice and access for too many families to the schools they want their children to attend. On the other hand, the policy also should and does recognize the interests of families who want to choose a walk zone school.
>
> Thus, I believe fifty percent walk zone preference provides a fair balance."

D Excerpts from Boston Policy Discussion

After a preliminary version of our research became available, Pathak and Sönmez interacted with BPS's staff. Parts of our research were presented to the Mayor's 27-member Executive Advisory Committee (EAC), which was charged with recommending amendments to the BPS school choice program. We explained that the BPS walk-zone priority was not having its intended impact because of the chosen precedence order. The EAC meeting minutes summarized the discussion (EAC 2013):

> "A committee member stated that the walk-zone priority in its current application does not have a significant impact on student assignment. The committee member noted that this finding was consistent with anecdotal evidence that the committee had heard from parents."

Following the presentation, the EAC immediately recommended that BPS switch to a "Compromise" precedence order, which first fills half of the walk-zone slots, then fills all the open slots, and then the fills the second half of the walk-zone slots. The Compromise precedence order attempts to even out the treatment of walk-zone applicants by changing the order of slots. Initially, when the first few open slots are processed, the walk-zone applicant pool has adversely selected lottery numbers, but this bias becomes less important by the time the last open slots are processed. The meeting minutes state:

> "BPS's recommendation is to utilize the [C]ompromise method in order to ensure that the walk-zone priority is not causing an unintended consequence that is not in stated policy."

Part of the Compromise method's initial appeal is the anticipated difficulty of describing a system that employs two lottery numbers. Switching to the Compromise treatment would increase the number of students assigned to their walk-zone schools. This change, together with the proposals to shrink zones or adopt a plan with smaller choice menus, raised concerns about decreased equity of access.

Our discovery about the role of precedence proved so significant that it became part of the fight between those favoring neighborhood assignment and those favoring increased choice. Proponents of neighborhood assignment interpreted our findings as showing that the (unintentional) improper implementation of the 50-50 school split caused hundreds of students to be shut out of their neighborhood schools. These proponents argued that changing the precedence order would be the only policy consistent with the School Committee's 1999 policy goals.

School choice proponents, on the other hand, seized on our findings for multiple reasons. Some groups, such as the activist Metropolitan Area Planning Council, fought fiercely to keep the 50-50 seat split with the existing precedence order (MAPC 2013):

> "The assignment priority given to walk-zone students has profound impacts on the outcomes of any new plan. The possible changes that have been proposed or discussed include increasing the set-aside, decreasing the set-aside, changing the processing order, or even reducing the allowable distance for walk zone priority to less than a mile. Actions that provide additional advantage to walk-zone students are likely to have a disproportionate negative impact on Black and Hispanic students, who are more reliant on out-of-walk-zone options for the quality schools in their basket."

The symbolism of the 50-50 split, combined with BPS's precedence order, resonated with sophisticated choice proponents by creating the inaccurate impression that they were somehow giving way to neighborhood proponents.

Confirming the counterintuitive nature of our results, other parties expressed skepticism as to how walk-zone priority as implemented did not have large implications for student assignment. For instance, the City Councillor in charge of education publicly testified (Connolly 2013):

> "MIT tells us that so many children in the walk zones of high demand schools 'flood the pool' of applicants, and that children in these walk zones get in in higher numbers, so walk zone priority doesn't really matter."
> "Maybe, that is true. But if removing the walk zone priority doesn't change anything, why change it all?"

In response to this and similar questions, we argued that moving away from the BPS priority/precedence structure would improve transparency and thus make it easier for BPS to adequately implement its policy goals.

Choice proponents also interpreted our findings as an argument for removing walk-zone priority entirely. Indeed, given that walk-zone priority (as implemented by BPS) plays only a small role in the outcome (relative to 0% Walk), simply eliminating it could increase transparency about how the system works. Getting rid of walk zone priority altogether avoids the (false) impression that applicants from the walk zone receive a boost under the mechanism.

In March 2013, the Boston school committee voted to adopt a new "Home-Based system," proposed by Peng Shi, under which each student receives an individualized choice menu based on his or her home address (Shi 2013). The new plan reduced the number of schools that applicants could rank but ensured that each applicant was able to rank a number of highly-rated schools.[36] Reducing the size of the choice menu under the Home-Based system, together with the subtle issues surrounding the implementation of the walk-zone reserve in Boston's historic 50-50 split, led Boston Superintendent Carol Johnson to support the idea of transparency. On March 13, 2013, Dr. Johnson announced (Johnson 2013):

> "After viewing the final MIT and BC presentations on the way the walk zone priority actually works, it seems to me that it would be unwise to add a second priority to the Home-Based model by allowing the walk zone priority be carried over."
>
> ...
>
> "Leaving the walk zone priority to continue as it currently operates is not a good option. We know from research that it does not make a significant difference the way it is applied today: although people may have thought that it did, the walk zone priority does not in fact actually help students attend schools closer to home. The External Advisory Committee suggested taking this important issue up in two years, but I believe we are ready to take this step now. We must ensure the Home-Based system works in an honest and transparent way from the very beginning."

The new plan went into effect for elementary and middle schools in Fall 2013.

E Boston Data Appendix

Relative to our two-priority-type model, BPS has three additional priority groups:

1. **guaranteed applicants**, who are typically continuing on at their current schools,
2. **sibling-walk applicants**, who have siblings currently attending a school and live in the walk zone, and
3. **sibling applicants**, who have siblings attending a school and live outside the walk zone.

Under BPS's priorities, applicants are ordered as follows:

[36] For additional research related to the new plan, see the work of Pathak and Shi (2014), Ashlagi and Shi (2015), and Shi (2015). For a popular account of the public policy debate surrounding assignment zones in Boston, see the article by Seelye (2013).

Walk-Zone Slots	Open Slots
Guaranteed	Guaranteed
Sibling-Walk	Sibling-Walk, Sibling
Sibling	
Walk	Walk, No Priority
No Priority	

A single random lottery number is used to order students within priority groups, and this number is the same for both types of slots.

We use data covering four years, from 2009–2012, when BPS employed a mechanism based on the student-proposing deferred acceptance algorithm. Each January, students interested in enrolling in or switching schools are asked to list schools for the first round. Students entering kindergarten can apply for elementary school at either Grade K1 or Grade K2 depending on whether they are four or five years old. Since the mechanism is based on the student-proposing deferred acceptance algorithm and there is no restriction on the number of schools that can be ranked, the assignment mechanism is strategy-proof.[37] BPS advises families on the application form:

> List your school choice in your true order of preference. If you list a popular school first, you won't hurt your chances of getting your second choice school if you don't get your first choice (BPS 2012).

Since the BPS mechanism is strategy-proof, we can isolate the effects of changes in priorities and precedence by holding submitted preferences fixed.[38]

In the Actual BPS policy (shown in Table 2), applicants with sibling priority who live outside the walk-zone apply to open slots before applying to walk-zone slots. Applicants with sibling priority who live in the walk-zone apply to walk zone slots before applying to open slots, as they would in Walk-Open.

[37]For analysis of the effects of restricting the number of choices that can be submitted, see the work of Haeringer and Klijn (2009), Calsamiglia, Haeringer, and Kljin (2010), and Pathak and Sönmez (2013).

[38]As a check on our understanding of the data, we verified that we can re-create the assignments produced by BPS. Across four years and three applicant grades, we can match 98% of the assignments. Based on discussions with BPS, we learned that the reason why we do not exactly re-create the BPS assignment is that we do not have access to BPS's exact capacity file, and instead must construct it *ex post* from the final assignment. There are small differences between this measure of capacity and the capacity input to the algorithm due to the handling of students who are administratively assigned. In our paper, to hold this feature fixed in our counterfactuals, we take our re-creation as representing the BPS assignment.

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