Games and Economic Behavior 69 (2010) 425–445

Contents lists available at ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

House allocation with existing tenants: A characterization ☆

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ARTICLE INFO

Article history:
Received 19 February 2007
Available online 30 October 2009

JEL classification:
C78
D70
D78

Keywords:
House allocation
Matching
Strategy-proofness
Consistency
Individual rationality
Pareto efficiency
Neutrality

ABSTRACT

We analyze mechanisms that are used to allocate dormitory rooms to students at college campuses. Students consist of newcoming freshmen, who do not currently occupy any rooms, and more senior students each of whom occupies a room from the previous year. In addition to the rooms already occupied by the existing tenants, there are vacated rooms by the graduating class. Students have strict preferences over dormitory rooms. Each student shall be assigned a dormitory room in an environment where monetary transfers are not allowed. An existing tenant can move to another room as a result of the assignment. We show that *you request my house–I get your turn* mechanisms are the only mechanisms that are *Pareto-efficient, individually rational, strategy-proof, weakly neutral*, and *consistent*.

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1. Introduction

Abdulkadiroğlu and Sönmez (1999) introduce *house allocation problems with existing tenants*: A set of houses should be allocated to a set of agents by a centralized clearinghouse. Some of the agents are existing tenants each of whom already occupies a house and the rest of the agents are newcomers. In addition to occupied houses, there are vacant houses. Existing tenants are not only entitled to keep their current houses but also apply for other houses.

This model is motivated by the real-life practices of on-campus housing at universities: The freshmen are the "newcomers", while the sophomores, juniors, and seniors are the "existing tenants". The rooms vacated by the graduating class are the "vacant" rooms, and the rooms already occupied in the previous year by the existing tenants are the "occupied" rooms.

An allocation is a *matching* of agents and houses so that each agent is assigned at most one house and no house is assigned to more than one agent. A *mechanism* is a systematic procedure that selects a matching for each problem.

Abdulkadiroğlu and Sönmez (1999) introduce the following mechanism which is referred to as the *You Request My House–I Get Your Turn* (*YRMH-IGYT*): Agents are prioritized in a queue and they are assigned their top choice house among still

☆ We would like to thank Fuhito Kojima, Marek Pycia, William Thomson, and participants at U. of British Columbia, Tilburg U., Waseda U., Tokyo U., Tokyo Tech U., NASMES'06 at U. of Minnesota, SWET'06 at Koç U., SCW'06 in Istanbul, CED'06 in Bodrum, Penn State U., and Texas A&M U. for comments. This version of the paper benefited extensively from the comments of an associate editor and an anonymous referee of the journal. We would like to thank NSF for support of our research. This paper supersedes the working paper entitled "Kidney Exchange with Good Samaritan Donors: A Characterization" (Sönmez and Ünver, 2006). All errors are our own responsibility.

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0899-8256/$ – see front matter © 2009 Published by Elsevier Inc.
doi:10.1016/j.geb.2009.10.010

unassigned houses in priority order. This continues until an agent "requests" the occupied house of an existing tenant who has not been assigned a house yet. In this case this request is put on hold; the existing tenant whose occupied house is requested is moved to the top of the queue, directly in front of the requester and the process continues with the modified queue. This is repeated any time there is a request for the occupied house of an existing tenant whose assignment is yet to be finalized. If a cycle of requests is formed, each existing tenant in the cycle is assigned the house she requested and removed from the system together with their assignments. Each priority order induces a different YRMH-IGYT mechanism.

Abdulkadiroğlu and Sönmez (1999) show that the YRMH-IGYT mechanisms are *Pareto-efficient, individually rational* (in the sense that each existing tenant is guaranteed a house that is no worse than her occupied house and each newcomer is guaranteed a house that is no worse than remaining unmatched option), and *strategy-proof*. In this paper, we present a full characterization of the YRMH-IGYT mechanisms based on these three axioms together with *weak neutrality* and *consistency*. *Weak neutrality* requires the outcome of a mechanism to be independent of the names or labels of the vacant houses. The formulation of *consistency* is less obvious in the present context. The traditional *consistency* axiom compares any pair of economies where one economy is obtained from the other by removal of a group of agents together with their assignments under the mechanism for which *consistency* is tested, and it requires this mechanism to insist on the same assignment as in the original economy for each remaining agent.[1] If a mechanism is *consistent*, then it eliminates incentives to renegotiate upon departure of any group of agents with their assignments. The difficulty, however, is that upon the removal of a group of agents with their assignments, what remains may not always be a well-defined economy. For example, if two existing tenants are assigned each others' occupied houses and if one of them leaves with her assignment, in what remains an existing tenant does not have her occupied house. A natural formulation here would be requiring *consistency* whenever the reduced economy is well-defined, but as it turns out this version is not strong enough for the full characterization of the YRMH-IGYT mechanisms.[2] For full characterization we also need a mechanism to insist on its outcome if a set of *unassigned* houses are removed (provided that what remains is a well-defined economy). The *consistency* axiom we present in the paper is the following: When a group of agents are removed from a problem together with their assignments under a mechanism ϕ and possibly together with some unassigned houses under ϕ, what remains may not be a well-defined problem. But if it is, then the assignments of the remaining agents under mechanism ϕ should not be affected by this departure. So if some of the exchanges are finalized while the others are pending, and even if some unassigned vacant houses become suddenly unavailable, the remaining agents should still have no reason to request another run of the mechanism.

House allocation with existing tenants model has another real-life application referred to as *kidney exchange*, which is recently brought to the attention of economists by Roth et al. (2004), Sönmez and Ünver (2006). In this problem, there are patients (similar to the "existing tenants") who would like to receive a compatible kidney, and their paired-donors (similar to the "occupied houses") whom they can trade in exchange of a better kidney donor. There are also options similar to the "vacant houses," such as altruistic donors who are not attached to any patient (see Sönmez and Ünver, 2006), or priority in the deceased donor waiting list (see Roth et al., 2004). On the other hand, there are no patients without attached paired-donors (similar to the "newcomers") in the way these papers formulate the problem.

Housing markets (Shapley and Scarf, 1974) and *house allocation problems* (Hylland and Zeckhauser, 1979) are two special cases of our model. Housing markets do not involve any vacant houses and newcomers. House allocation problems do not involve any existing tenants and occupied houses. YRMH-IGYT mechanism is a generalization of both *core* mechanism for housing markets and the *simple serial dictatorship* for house allocation. When preferences are strict, there is a unique core outcome of a housing market (Roth and Postlewaite, 1977) which can be determined through *Gale's top trading cycles* algorithm (attributed to David Gale by Shapley and Scarf, 1974). Moreover, in this case it is also *strategy-proof* (Roth, 1982). Indeed it is the only mechanism that is *Pareto-efficient, individually rational*, and *strategy-proof* (Ma, 1994). In the context of housing markets, Svensson (1999) shows that the *simple serial dictatorship* is the only mechanism that is *strategy-proof, nonbossy*, and *neutral* while Ergin (2000) shows it is the only mechanism that is *Pareto-efficient, consistent*, and *neutral*. Our characterization is a natural generalization of each of Ma (1994), Svensson (1999), and Ergin (2000) results.[3]

In two related papers, Pápai (2000) characterizes *group strategy-proof, Pareto-efficient*, and *reallocation-proof* mechanisms, and Pycia and Ünver (2007) characterize *group strategy-proof* and *Pareto-efficient* mechanisms in house allocation economies. Although these economies do not have any individual endowments (unlike in a house allocation problem with existing tenants), the characterized mechanisms by these papers, called *Hierarchical Exchange* mechanisms and *Trading Cycles with Brokers and Owners*, respectively, mimic trading procedures with individual endowments, which are induced according to an inheritance structure. Though YRMH-IGYT mechanisms are in the class of hierarchical exchange mechanisms and also in the class of trading cycles with brokers and owners mechanisms, in general, these two latter classes fail to satisfy *individual rationality*, *weak neutrality*, and *consistency*.

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[1] See Thomson (1996) for a comprehensive survey.

[2] The mechanism in Example 7 satisfies all four other axioms and this version of *consistency* but not the stronger version we present in the paper.

[3] Other axiomatic studies in housing markets and house allocation include Chambers (2004), Ehlers (2002), Ehlers and Klaus (2007), Ehlers et al. (2002), Kesten (2009), Miyagawa (2002), Pápai (2007), Velez-Cordona (2006).

2. House allocation problem with existing tenants

Let $\mathcal{I}$ be a finite set of agents and $\mathcal{H}$ be a finite set of houses. We refer to $\mathcal{I}$ as the **potential set of agents** and $\mathcal{H}$ as the **potential set of houses**. Set $\mathcal{I}$ is partitioned as $\{\mathcal{I}_\mathcal{E}, \mathcal{I}_\mathcal{N}\}$. Set $\mathcal{I}_\mathcal{E}$ is referred to as the **potential existing tenant set** and $\mathcal{I}_\mathcal{N}$ is referred to as the **potential newcomer set**. Each agent $i \in \mathcal{I}_\mathcal{E}$ has a paired house $h_i \in \mathcal{H}$. We refer to h_i as the **occupied house of potential existing tenant i**. No two potential existing tenants have the same occupied house, that is $h_i = h_j \Longrightarrow i = j$ for any $i, j \in \mathcal{I}_\mathcal{E}$. We assume that $|\mathcal{H}| > |\mathcal{I}_\mathcal{E}|$, that is, there exists at least one house that is not occupied by a potential existing tenant. Each $i \in \mathcal{I}_\mathcal{E}$ has strict preferences P_i on all houses in $\mathcal{H}$ and the remaining unmatched option denoted by h_0 such that $h_i P_i h_0$, that is, each potential existing tenant prefers her occupied house over the remaining unmatched option h_0. Each potential newcomer $i \in \mathcal{I}_\mathcal{N}$ has strict preferences over $\mathcal{H}$ and the remaining unmatched option h_0. For each agent, let R_i denote the weak preference relation induced by P_i. For any agent $i \in \mathcal{I}$ and any subset of houses $H \subseteq \mathcal{H}$, let $\mathcal{R}(i, H)$ denote the set of all strict preferences over $H \cup \{h_0\}$ such that if $i \in \mathcal{I}_\mathcal{E}$ and $h_i \in H$ then $h_i P_i h_0$. Let $\mathcal{R}(I, H)$ denote the set of all feasible strict preference profiles over $H \cup \{h_0\}$ for agents in I, that is $\mathcal{R}(I, H) = \prod_{i \in I} \mathcal{R}(i, H)$.

A **house allocation problem with existing tenants**, or simply a **problem**, is a list $\langle I, H, R\rangle$ where:

- $I \subseteq \mathcal{I}$ is a set of agents,
- $H \subseteq \mathcal{H}$ is a set of houses such that for all $i \in \mathcal{I}_\mathcal{E} \cap I$, $h_i \in H$, and
- $R = (R_i)_{i \in I} \in \mathcal{R}(I, H)$ is a preference profile.

Given a problem $\langle I, H, R\rangle$, we partition I as $\{I_E, I_N\}$ and H as $\{H_O, H_V\}$. Set $I_E = \mathcal{I}_\mathcal{E} \cap I$ is the **set of existing tenants.** Set $H_O = \{h_i\}_{i \in I_E}$ is the set of houses occupied by the existing tenants, and we refer to it as the **set of occupied houses**. Set $I_N = I \setminus I_E = \mathcal{I}_\mathcal{N} \cap I$ is the **set of newcomers**. Set $H_V = H \setminus H_O$ is the **set of vacant houses**. Note that the occupied house h_j of a potential existing tenant $j \in \mathcal{I}_\mathcal{E}$ is formally a vacant house in a problem $\langle I, H, R\rangle$ if $j \notin I$ and $h_j \in H$.[4]

Since the information on existing tenants, newcomers, occupied houses, and vacant houses is embedded in a preference profile, whenever convenient, we will denote a problem with simply a preference profile.

Given $I \subseteq \mathcal{I}$ and $H \subseteq \mathcal{H}$, a **matching** is a mapping $\mu : I \to H \cup \{h_0\}$ such that

$$\mu(i) \neq \mu(j) \quad \text{or} \quad \mu(i) = \mu(j) = h_0 \quad \text{for any distinct } i, j \in I.$$

We refer to $\mu(i)$ as the **assignment** of agent i. A matching is simply an assignment of houses to agents such that each agent is assigned a distinct house from the rest of the agents or unmatched. Let $\mathcal{M}(I, H)$ denote the set of matchings for given I, H.

A **mechanism** is a systematic procedure that assigns a matching for each problem R. The outcome of mechanism ϕ for problem R is denoted by $\phi[R]$ and the assignment of agent i under ϕ for problem R is denoted by $\phi[R](i)$. For any $J \subseteq I$, let $\phi[R](J) = \{\phi[R](j)\}_{j \in J}$ be the set of houses assigned to agents in J.

3. The axioms

3.1. Individual rationality, Pareto efficiency, and strategy-proofness

Throughout this section, we fix the set of agents $I \subseteq \mathcal{I}$ and the set of houses $H \subseteq \mathcal{H}$ as defined above.

A matching is ***individually rational*** if no existing tenant is assigned a house worse than her occupied house and no newcomer is assigned a house worse than remaining unmatched. Formally, a matching $\mu \in \mathcal{M}$ is *individually rational*, if $\mu(i) R_i h_i$ for any $i \in I_E$ and $\mu(i) R_i h_0$ for any $i \in I_N$. A mechanism is ***individually rational*** if it always selects an individually rational matching.

A matching is ***Pareto-efficient*** if there is no other matching that makes every agent weakly better off and some agent strictly better off. Formally, a matching $\mu \in \mathcal{M}$ is *Pareto-efficient* if there is no matching $\nu \in \mathcal{M}$ such that $\nu(i) R_i \mu(i)$ for all $i \in I$ and $\nu(j) P_j \mu(j)$ for some $j \in I$. A mechanism is ***Pareto-efficient*** if it always selects a Pareto-efficient matching.

A mechanism is ***strategy-proof*** if no agent can ever benefit by misrepresenting her preferences. Formally a mechanism ϕ is *strategy-proof* if for any problem $R \in \mathcal{R}(I, H)$, any agent $i \in I$, and any potential misrepresentation $R_i^* \in \mathcal{R}(i, H)$, we have $\phi[R_i, R_{-i}](i) R_i \phi[R_i^*, R_{-i}](i)$.

3.2. Weak neutrality and consistency

Each of the three axioms we introduced so far is defined for fixed sets of agents and houses. In contrast, our next axiom *weak neutrality* relates problems with possibly different sets of houses and final axiom *consistency* relates problems with different sets of agents and houses.

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[4] This observation will be useful when we formalize the consistency axiom later on. We will be considering such situations as existing tenant j being assigned a vacant house and leaving the problem. The occupied house h_j of existing tenant j is no longer attached to any agent in the reduced problem, and hence treated as a vacant house.

A mechanism is ***weakly neutral*** if labeling of vacant houses has no effect on the outcome of the mechanism. We need additional notation to define weak neutrality formally. For any $I_E \subseteq \mathcal{I}_{\mathcal{E}}$, a **permutation of vacant houses for** I_E is a one-to-one and onto function $\pi : \mathcal{H} \cup \{h_0\} \to \mathcal{H} \cup \{h_0\}$ such that $\pi(h_i) = h_i$ for any $i \in I_E$ and $\pi(h_0) = h_0$. For any $h \in \mathcal{H} \cup \{h_0\}$, we refer to $\pi(h)$ as the **label of** h **under** π. Observe that only vacant houses are relabeled under π. For any problem $\langle I, H, R\rangle$, any permutation π of vacant houses for I_E, and any $i \in I$, let $R_i^\pi \in \mathcal{R}(i, \{\pi(h)\}_{h\in H})$ be such that

$$gR_i^\pi h \iff \pi^{-1}(g) R_i \pi^{-1}(h) \quad \text{for any } g, h \in \{\pi(e)\}_{e\in H\cup\{h_0\}}.$$

Let $R^\pi = (R_i^\pi)_{i\in I}$. Formally, a mechanism ϕ is *weakly neutral* if for any problem $\langle I, H, R\rangle$ and any permutation π of vacant houses for I_E, we have

$$\phi[R^\pi](i) = \pi(\phi[R](i)) \quad \text{for any } i \in I.$$

We need additional notation to introduce our final axiom.

For any agent $i \in I$, preference relation $R_i \in \mathcal{R}(i, H)$, and set of houses $G \subset H$, let $R_i^G \in \mathcal{R}(i, G)$ be the **restriction of preference** R_i **to houses in** G. That is,

$$gR_i^G h \iff gR_i h \quad \text{for any } g, h \in G \cup \{h_0\}.$$

For any $J \subset I$, let $R_J = (R_i)_{i\in J}$ be the **restriction of profile** R **to agents in** J. Given fixed $I \subseteq \mathcal{I}$ and $H \subseteq \mathcal{H}$ and $R \in \mathcal{R}(I, H)$, we will often denote $R_{I\setminus J}$ by R_{-J}. For any $J \subset I$ and $G \subset H$, let $R_J^G = (R_i^G)_{i\in J}$ be the **restriction of profile** R **to agents in** J **and houses in** G. Given fixed $H \subseteq \mathcal{H}$, $R_i \in \mathcal{R}(i, H)$, we will often denote $R_i^{H\setminus G}$ by R_i^{-G}.

Given a problem $\langle I, H, R\rangle$, sets of agents $J \subset I$, and sets of houses $G \subset H$ we refer to $\langle J, G, R_J^G\rangle$ as the **restriction of problem** $\langle I, H, R\rangle$ **to agents in** J **and houses in** G. The triple $\langle J, G, R_J^G\rangle$ itself is a **well-defined reduced problem** if $G_O = \{h_j\}_{j\in J_E}$, that is, the occupied houses of existing tenants in J is the set of occupied houses in G.

Given a problem $\langle I, H, R\rangle$, the removal of a set of agents $J \subset I$ together with their assignments $\phi[R](J)$ and some unassigned houses $G \subset H$ under ϕ results in a well-defined reduced problem $\langle I \setminus J, H \setminus (\phi[R](J) \cup G), R_{-J}^{-\phi[R](J)\cup G}\rangle$ if $(\phi[R](J) \cup G) \cap \{h_i\}_{i\in I_E\setminus J} = \emptyset$.

A mechanism ϕ is ***consistent*** if for any problem $\langle I, H, R\rangle$, whenever the removal of a set of agents $J \subset I$ together with their assignments $\phi[R](J)$ and some unassigned houses $G \subset H$ results in a well-defined reduced problem, then

$$\phi[R_{-J}^{-\phi[R](J)\cup G}](i) = \phi[R](i) \quad \text{for any } i \in I \setminus J.$$

So under a *consistent* mechanism, the removal of

- a set of agents,
- their assignments, and
- a set of unassigned houses

does not affect the assignments of remaining agents provided that the removal results in a well-defined reduced problem.[5]

4. You Request My House–I Get Your Turn mechanism

You Request My House–I Get Your Turn mechanism (or *YRMH-IGYT* mechanism in short) is introduced by Abdulkadiroğlu and Sönmez (1999) and further studied by Chen and Sönmez (2002) and Sönmez and Ünver (2006).[6] In order to define this mechanism we need the following additional notation:

A **(priority) ordering** is a one-to-one and onto function $f : \{1, 2, \ldots, |\mathcal{I}|\} \to \mathcal{I}$. Here $f(1)$ indicates the agent with the highest priority in $\mathcal{I}$, $f(2)$ indicates the agent with the second highest priority in $\mathcal{I}$, and so on. Let $\mathcal{F}$ be the set of all orderings. Given a set of agents $J \subseteq \mathcal{I}$, agent $j \in J$ is the **highest priority agent in** J **under** f if $f^{-1}(j) \leqslant f^{-1}(i)$ for any $i \in J$. Given a set of agents $J \subseteq \mathcal{I}$, the **restriction of** f **to** J is an ordering f_J of the agents in J which orders them as they are ordered in f. Formally $f_J : \{1, 2, \ldots, |J|\} \to J$ is a one-to-one and onto function such that for any $i, j \in J$,

$$f_J^{-1}(i) \leqslant f_J^{-1}(j) \iff f^{-1}(i) \leqslant f^{-1}(j).$$

Each ordering $f \in \mathcal{F}$ defines a YRMH-IGYT mechanism. Let ψ^f denote the YRMH-IGYT mechanism induced by ordering $f \in \mathcal{F}$. For any problem $\langle I, H, R\rangle$, let $\psi^f[R]$ denote the outcome of the YRMH-IGYT mechanism induced by ordering f_I for this problem.

For any problem $\langle I, H, R\rangle$, matching $\psi^f[R]$ is obtained with the following YRMH-IGYT algorithm in several rounds.

[5] Consistency for house allocation problem with existing tenants is in the same spirit as consistency defined for exchange economies by Thomson (1992) and Dagan (1995). Thomson defines consistency in generalized economies with social and private endowments. Dagan considers Walrasian economies, but allows the solutions to be empty-valued.

[6] Abdulkadiroğlu and Sönmez (1999) provided two algorithms, *You Request My House–I Get Your Turn (YRMH-IGYT)* algorithm and the *Top Trading Cycles (TTC)* algorithm, to implement this mechanism. The description we provide below is based on the description that utilizes the TTC algorithm.

Round 1. Construct a graph in which each agent, each house, and remaining unmatched option h_0 is a node. In this graph:

- each agent "points to" her top choice, a house or option h_0 (i.e. there is a directed link from each agent to her top choice),
- each occupied house $h_i \in H_O$ points to its tenant i,
- each vacant house points to the highest priority agent in I under f, and
- option h_0 points to all newcomers.

Since there is a finite number of agents and houses, there is at least one cycle. No agent or house is in more than one cycle, however remaining unmatched option can be in more than one cycle. (A **cycle** is either (i) a list $(g_1, j_1, \ldots, g_k, j_k)$ of houses and agents where house g_1 points to agent j_1, agent j_1 points to house g_2, house g_2 points to agent j_2, ..., house g_k points to agent j_k, and agent j_k points to house g_1, or (ii) a list (h_0, i) where option h_0 points to agent i and agent i points to h_0.) Assign each agent in each cycle the choice she points to and remove each such cycle from the graph.

In general, at

Round t. Construct a new graph with the remaining agents, houses, and the remaining unmatched option h_0 such that

- each remaining agent points to her first choice among the remaining houses and option h_0,
- each remaining occupied house $h_i \in H_O$ points to its tenant i in case its tenant i remains in the problem, and to the highest priority remaining agent under f otherwise,
- each remaining vacant house points to the highest priority remaining agent under f, and
- option h_0 points to all remaining newcomers.

There is at least one cycle. No agent and house is in more than one cycle, though option h_0 can be in more than one cycle. Carry out the implied exchange in each cycle.

The algorithm terminates when there is no agent left in the graph.
We demonstrate the execution of the algorithm with an example:

Example 1. Let $I_E = \{i_1, i_2, i_3, i_4\}$, $I_N = \{i_5, i_6, i_7\}$, $H_O = \{h_1, h_2, h_3, h_4\}$, $H_V = \{d, e, g\}$. Let $f = (i_5, i_1, i_3, i_7, i_6, i_2, i_4)$ be the ordering of the agents. The preferences of the agents are given as follows:

Agent 1: $h_2 \; P_1 \cdots$

Agent 2: $h_1 \; P_2 \cdots$

Agent 3: $d \; P_3 \; h_4 \; P_4 \cdots$

Agent 4: $e \; P_4 \cdots$

Agent 5: $d \; P_5 \cdots$

Agent 6: $h_0 \; P_6 \cdots$

Agent 7: $e \; P_7 \; h_3 \; P_7 \cdots$

The outcome of the YRMH-IGYT mechanism is found as follows:

Round 1. Each agent points to her first choice. Each occupied house points to its existing tenant. Each vacant house points to $f(1) = i_5$. Option h_0 points to all newcomers. The resulting graph has three cycles, (d, i_5), (h_1, i_1, h_2, i_2), and (h_0, i_6) (see Fig. 1).

We remove them from the problem by assigning each agent in each cycle the option she is pointing to:

$$\psi^f(i_1) = h_2, \qquad \psi^f(i_2) = h_1, \qquad \psi^f(i_5) = d, \qquad \psi^f(i_6) = h_0.$$

Round 2. Each remaining agent points to her first remaining choice. Each remaining occupied house points to its tenant, since its tenant is still in the problem. Each remaining vacant house points to the highest priority remaining agent, $f(3) = i_3$. Option h_0 points to remaining newcomers. The resulting graph has a single cycle (e, i_3, h_4, i_4) (see Fig. 2). We remove it from the problem by assigning each agent in the cycle the option she is pointing to:

$$\psi^f(i_3) = h_4, \qquad \psi^f(i_4) = e.$$

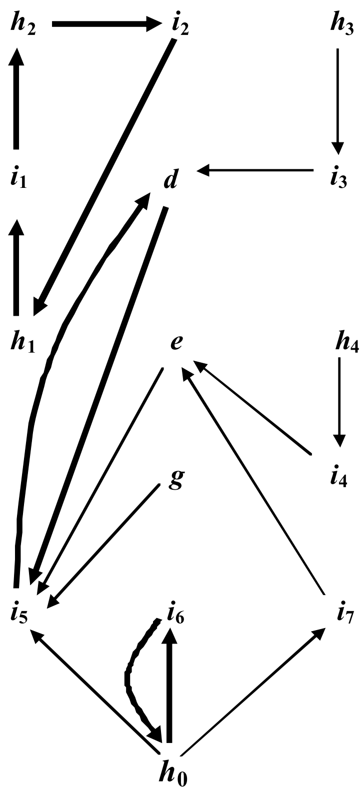

Fig. 1. Round 1 of Example 1.

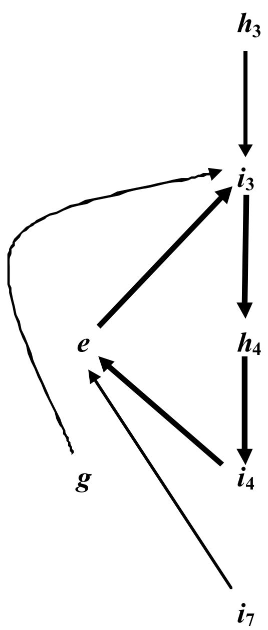

Fig. 2. Round 2 of Example 2.

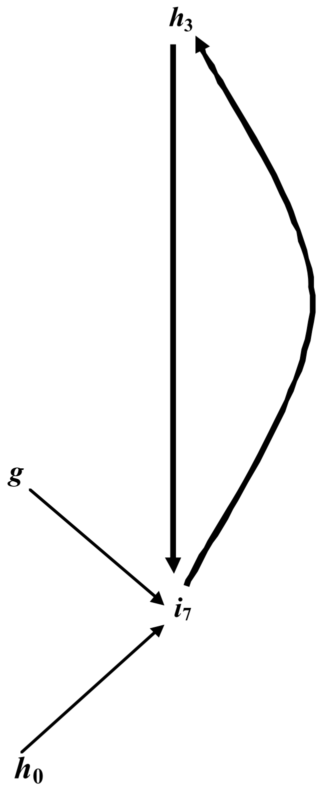

Fig. 3. Round 3 of Example 1.

Round 3. Only one agent is left and she is a newcomer. All houses and option h_0 point to her, while she points to her first choice remaining, house h_3. We obtain a graph with the cycle (h_3, i_7) (see Fig. 3). We remove it from the problem and set

$$\psi^f(i_7) = h_3.$$

The procedure is terminated, since all agents are assigned either a house or option h_0.

5. Characterization of the YRMH-IGYT mechanisms

Our main result is a characterization of the YRMH-IGYT mechanism:

Theorem 1. *A mechanism is* Pareto-efficient, individually rational, strategy-proof, weakly neutral, *and* consistent *if and only if it is a YRMH-IGYT mechanism.*

We present our main result through two propositions:

Proposition 1. *For any ordering $f \in \mathcal{F}$, the induced YRMH-IGYT mechanism ψ^f is* Pareto-efficient, individually rational, strategy-proof, weakly neutral, *and* consistent.

Proposition 2. *Let ϕ be a* Pareto-efficient, individually rational, strategy-proof, weakly neutral, *and* consistent *mechanism. Then $\phi = \psi^f$ for some $f \in \mathcal{F}$.*

We prove these propositions in Appendix A.

The interpretation of Proposition 1 is straightforward as a mechanism design result. Interpretation of Proposition 2 is trickier. Proposition 2 says that given any mechanism that satisfies the aforementioned properties, we can find a YRMH-IGYT mechanism that generates the same matching as the original mechanism does for each problem, thus, these two mechanisms are equivalent. Hence, this proposition is, in a way, an *implementation* result. Suppose that as a mechanism designer, we would like to fully describe a given mechanism satisfying the aforementioned properties. The mechanism is a black box for us and all we are able to do is to execute the black box to find its outcome for any given problem. Thus, for us, this mechanism is a function that assigns a matching for each problem. But, this description will be demanding in terms of information processing, execution time, and information storage space, if the domain of problems is large. On the other hand, using the proof of Proposition 2, we can construct an ordering of agents by finding the outcome of the mechanism for only $|\mathcal{I}|-1$ problems, instead of the whole domain (which has an exponentially increasing number of preference profiles as $|\mathcal{I}|$ increases), and use the induced YRMH-IGYT mechanism to fully characterize this particular mechanism. Therefore, Proposition 2 gives us a less demanding way of describing such a mechanism through a YRMH-IGYT mechanism, which has the desired properties as shown by Proposition 1. This feature can be appealing in implementing real-life market mechanisms.

6. Independence of the axioms

The following examples establish the independence of the axioms.

Example 2. Fix an ordering $f \in \mathcal{F}$. For each problem $\langle I, H, R\rangle$, let mechanism ϕ assign each agent $i \in I_E$ her occupied house h_i, and the vacant houses are distributed to the newcomers according to the *serial dictatorship induced by* f: the highest priority agent in I_N is assigned her top choice in $H_V \cup \{h_0\}$, the second highest priority agent is assigned her top choice among the remaining vacant houses in H_V and option h_0, etc.

Mechanism ϕ is *individually rational*, *strategy-proof*, *weakly neutral*, and *consistent* but not *Pareto-efficient*.

Example 3. Fix an ordering $f \in \mathcal{F}$ and let mechanism ϕ be the *serial dictatorship induced by* f: For any problem $\langle I, H, R\rangle$, the highest priority agent in I is assigned her top choice in $H \cup \{h_0\}$, the second highest priority agent is assigned her top choice among remaining houses and option h_0, etc.

Mechanism ϕ is *Pareto-efficient*, *strategy-proof*, *weakly neutral*, and *consistent* but not *individually rational*.

Example 4. Fix an ordering $f \in \mathcal{F}$. Let $f(1)$ be a potential existing tenant. Let $g \in \mathcal{F}$ be constructed from f by demoting agent $f(1)$ to the very end of the ordering (so that the highest priority agent under f is the lowest priority agent under g) but otherwise keeping the rest of the priority ordering as in f. For any problem $\langle I, H, R\rangle$, let

$$\phi[R] = \begin{cases} \psi^g[R] & \text{if } f(1) \in I_E,\ hR_ih_{f(1)} \text{ for all } i \in I, \text{ and } h \in H, \\ \psi^f[R] & \text{otherwise.} \end{cases}$$

That is, mechanism ϕ picks the outcome of the YRMH-IGYT mechanism induced by ordering g if each agent (including agent $f(1)$) ranks the occupied house of agent $f(1)$ as her last choice, and picks the outcome of the YRMH-IGYT mechanism induced by ordering f otherwise.

Mechanism ϕ is *Pareto-efficient*, *individually rational*, *weakly neutral*, and *consistent* but not *strategy-proof*.

Example 5. Let $\mathcal{I}$ and $\mathcal{H}$ be such that $|\mathcal{I}| \geqslant 2$. Let $i_1, i_2 \in \mathcal{I}$ be distinct agents and $h^* \in \mathcal{H} \setminus \{h_i\}_{i \in \mathcal{I}_\mathcal{E}}$. Let $f, g \in \mathcal{F}$ be such that $f(1) = g(2) = i_1$, $f(2) = g(1) = i_2$ and $f(i) = g(i)$ for all $i \in \mathcal{I} \setminus \{i_1, i_2\}$. For any problem $\langle I, H, R\rangle$, let

$$\phi[R] = \begin{cases} \psi^f[R] & \text{if } i_1 \in I,\ h^* \in H_V \text{ and } h^* R_{i_1} h \text{ for all } h \in H_V, \\ \psi^g[R] & \text{otherwise.} \end{cases}$$

That is, mechanism ϕ picks the outcome of the YRMH-IGYT mechanism induced by ordering f if both agent i and vacant house h^* are present, and agent i_1 prefers vacant house h^* to any other vacant house, and mechanism ϕ picks the outcome of the YRMH-IGYT mechanism induced by ordering g otherwise.

Mechanism ϕ is *Pareto-efficient*, *individually rational*, *strategy-proof*, and *consistent* but not *weakly neutral*.

Example 6. Let $f, g \in \mathcal{F}$ be such that $f \neq g$ and $f(1) = g(1) = i$ for some $i \in \mathcal{I}_\mathcal{E}$. For any problem $\langle I, H, R\rangle$, two cases are possible:

- $i \notin I_E$: Then $\phi[R] = \psi^f[R]$.
- $i \in I_E$: Then let $\phi[R]$ be the outcome of the hierarchical exchange mechanism (Pápai, 2000) with the following inheritence rule:
  - All vacant houses in H_V are initially inherited by agent i. During the execution of the hierarchical exchange algorithm, if i is matched with h_i, then all vacant houses are inherited according to the priority order g (i.e., vacant house h is first inherited by agent $g(2)$; when $g(2)$ is matched, it is inherited by $g(3)$, so on so forth); otherwise, they are inherited according to priority order f in further rounds of the algorithm.
  - For each $j \in I_E$, occupied house h_j is initially inherited by agent j. Once j is matched and h_j is not, h_j is inherited according to the rule of the vacant houses specified above in further rounds of the algorithm.

Mechanism ϕ is *Pareto-efficient*, *individually rational*, *strategy-proof*, and *weakly neutral* but not *consistent*.

Example 7. Another example regarding the consistency axiom is as follows: Let $f, g \in \mathcal{F}$ be such that $f \neq g$. For any problem $\langle I, H, R\rangle$, let

$$\phi[R] = \begin{cases} \psi^f[R] & \text{if there are odd number of vacant houses,} \\ \psi^g[R] & \text{if there are even number of vacant houses.} \end{cases}$$

Mechanism ϕ is *Pareto-efficient*, *individually rational*, *strategy-proof*, and *weakly neutral*, but not *consistent*.

Appendix A. Proofs of the results

Before, we prove our results, we state the following modification of the YRMH-IGYT algorithm. We will use this version of the algorithm in our proofs. Since a cycle remains as a cycle in the next round if it is not removed in the previous round in the algorithm, both versions are equivalent to each other. Let $f \in \mathcal{F}$:

Round 1(a). Construct a graph in which each agent, each house and option h_0 is a node. In this graph:

- each agent "points to" her top choice, a house or option h_0,
- each occupied house $h_i \in H_O$ points to its tenant i,
- each vacant house points to the agent with the highest priority in I under f, and
- option h_0 points to all newcomers.

Since there is a finite number of agents and houses, there is at least one cycle. If each cycle *includes a vacant house* then skip to Round 1(b). Otherwise consider each cycle without a vacant house. Assign each agent in such a cycle the option she points to, a house or option h_0, and remove each such cycle from the graph. Construct a new graph with the remaining agents, houses and option h_0 such that

- each remaining agent points to her first choice among the remaining houses and option h_0,
- each remaining occupied house $h_i \in H_O$ points to its tenant i,
- each vacant house points to the highest priority remaining agent under f, and
- option h_0 points to all remaining newcomers.

There is a cycle. If each cycle *includes a vacant house* then skip to Round 1(b); otherwise carry out the implied exchange in each such cycle and proceed similarly until either no agent is left or each remaining cycle includes a vacant house.

Round 1(b). Since each vacant house points to the highest priority agent among remaining agents under f, there is a unique cycle in the graph, and it includes both the highest priority agent among remaining agents and a unique vacant house. Assign each agent in such a cycle the house she points to and remove each such cycle from the graph. Proceed with Round 2.

In general, at

Round t(a). Construct a new graph with the remaining agents, houses and option h_0 such that

- each remaining agent points to her first choice among the remaining houses and option h_0,
- each remaining occupied house $h_i \in H_O$ points to its tenant i in case its tenant i remains in the problem, and to the highest priority remaining agent under f otherwise,
- each remaining vacant house points to the highest priority remaining agent under f, and
- option h_0 points to all remaining newcomers.

There is a cycle. If the only remaining cycle includes either a vacant house or an occupied house whose tenant has left, then skip to Round t(b); otherwise carry out the implied exchange in each such cycle and proceed similarly until either no agent is left or the only remaining cycle includes either a vacant house or an occupied house whose tenant has left.

Round t(b). There is a unique cycle in the graph, and it includes the highest priority agent among remaining agents under f and either a vacant house or an occupied house whose tenant has left. Assign each agent in such a cycle the house she points to and remove each such cycle from the graph. Proceed with Round t+1.

The algorithm terminates when there is no agent left in the graph.

In the rest of the paper, when we talk about "the YRMH-IGYT algorithm" or "the algorithm", we will be referring to the above modified version.

Proof of Proposition 1. Let $f \in \mathcal{F}$. *Pareto efficiency*, *individual rationality*, and *strategy-proofness* of ψ^f follows from Abdulkadiroğlu and Sönmez (1999). *Weak neutrality* of ψ^f directly follows from the description of the YRMH-IGYT algorithm (i.e., under the relabeled economy, the relabeled version of the same sequence of cycles will form).

We next prove that ψ^f is *consistent*. Fix a problem $\langle I, H, R\rangle$. Let $J \subset I$ be such that $\psi^f[R](J) \cap \{h_i\}_{i \in I_E \setminus J} = \emptyset$ and $G \subset H$ so that the reduced problem $\langle I \setminus J, H \setminus (\psi^f[R](J) \cup G), R_{-J}^{-\psi^f[R](J)\cup G}\rangle$ is well-defined. Consider the execution of the YRMH-IGYT algorithm to obtain matching $\psi^f[R]$ and suppose it terminates after round t^*. For any $t \in \{1, 2, \ldots, t^*\}$, let A^t be the set of agents who formed cycles and received their assignments in Round t(a), and let B^t be the set of agents who formed a cycle and received their assignments in Round t(b). Since no agent in J is assigned the occupied house of an agent in $I_E \setminus J$, set J can be partitioned as $\{I^t, J^t\}_{t \in \{1,2,\ldots,t^*\}}$ where

- $I^t \subseteq A^t$ is a set of agents who form one or more cycles in Round t(a) of YRMH-IGYT algorithm, and
- $J^t \subseteq B^t$ is a set of agents $\{j_1, j_2, \ldots, j_k\}$ such that
  1. $\psi^f[R](j_\ell) = h_{j_{\ell+1}}$ for any $\ell \in \{1, 2, \ldots, k-1\}$, and
  2. $\psi^f[R](j_k)$ is a vacant house or the occupied house of an existing tenant in $\bigcup_{s=1}^{t-1} J^s$.

Consider, the reduced problem $R_{-J}^{-\psi^f[R](J)\cup G}$, and the execution of YRMH-IGYT algorithm to obtain $\psi^f[R_{-J}^{-\psi^f[R](J)\cup G}]$.

Round 1(a): In Round 1(a), having removed the agents in J has no affect on any remaining cycles and all agents in $A^1 \setminus I^1$ form the same cycles as in the original problem. Since some of the houses in the original problem are removed in the reduced problem, cycles that form in subsequent rounds in the original problem may form earlier in Round 1(a) in the reduced problem. A cycle that is not removed remains a cycle in subsequent rounds until removed. Keep any cycle involving agents in $I \setminus (A^1 \cup B^1)$ until the round it formed under the original problem and skip to Round 1(b).

Round 1(b): If $J^1 = B^1$, then the exact same cycle forms in Round 1(b) as before and each agent in B^1 receives the same assignment as before. If $J^1 = \emptyset$ then this round is skipped. Let $J^1 \subset B^1$ be such that $J^1 \neq \emptyset$. Let $(h_v, i_1, h_{i_2}, i_2, \ldots, h_{i_k}, i_k)$ be the cycle formed in Round 1(b) of the original problem where i_1 is the highest priority agent in $I \setminus A^1$ under ordering f and h_v is a vacant house. We have $J^1 = \{i_\ell, i_{\ell+1}, \ldots, i_k\}$ for some $\ell \in \{2, \ldots, k\}$ for otherwise someone in J_1 would have been assigned the occupied house of an existing tenant who has been removed (and thus the reduced problem would not have been well-defined). Having been the highest priority agent in a larger set, agent i_1 is still the highest priority agent among the remaining agents. Moreover since agent i_ℓ has been removed, house h_{i_ℓ} is a vacant house in the reduced problem. Hence house h_{i_ℓ} points to i_1 in Round 1(b). In addition agent i_1 points to h_{i_2} (as before), house h_{i_2} points to agent i_2 (as before), ..., agent $i_{\ell-1}$ points to h_{i_ℓ} (as before). Hence $(h_{i_\ell}, i_1, h_{i_2}, \ldots, h_{i_{\ell-1}}, i_{\ell-1})$ is a cycle in Round 1(b). Therefore each agents in $B^1 \setminus J^1$ receives the same assignment in the reduced problem as before. We remove this cycle from the reduced problem and proceed with Round 2.

We similarly continue with Round 2, and so on.[7] Therefore, each agent in $I \setminus J$ is assigned the same house as under $\psi^f[R]$, completing the proof. □

Proof of Proposition 2. Let ϕ be a *Pareto-efficient*, *individually rational*, *strategy-proof*, *weakly neutral*, and *consistent* mechanism. Fix $h^* \in \mathcal{H} \setminus \{h_i\}_{i \in \mathcal{I}_\mathcal{E}}$. We will recursively construct an ordering $f \in \mathcal{F}$ as follows:

- We determine $f(1)$ as follows: Let $R^1 \in \mathcal{R}(\mathcal{I}, \mathcal{H})$ be such that for any $i \in \mathcal{I}_\mathcal{E}$,

  $$h^* P_i^1 h_i P_i^1 h \quad \text{for any } h \in \left(\mathcal{H} \setminus \left\{h^*, h_i\right\}\right) \cup \{h_0\} \quad \text{and}$$

  for any $i \in \mathcal{I}_\mathcal{N}$,

  $$h^* P_i^1 h_0 P_i^1 h \quad \text{for any } h \in \mathcal{H} \setminus \left\{h^*\right\}.$$

  By *Pareto efficiency* of ϕ, there exists some $i_1 \in \mathcal{I}$ such that $\phi[R^1](i_1) = h^*$. Let $f(1) = i_1$. Moreover, by *individual rationality* of ϕ, $\phi[R^1](i) = h_i$ for all $i \in \mathcal{I}_\mathcal{E} \setminus \{i_1\}$ and $\phi[R^1](i) = h_0$ for all $i \in \mathcal{I}_\mathcal{N} \setminus \{i_1\}$.
- For any $t > 1$, upon determining agents $f(1), f(2), \ldots, f(t-1)$, we determine $f(t)$ as follows: Let $R^t \in \mathcal{R}(\mathcal{I}, \mathcal{H})$ be such that
  * $R_i^t = R_i^1$ for any $i \in \mathcal{I} \setminus \{f(1), f(2), \ldots, f(t-1)\}$,
  * $h_0 P_i^t h$ for any $i \in \{f(1), f(2), \ldots, f(t-1)\} \cap \mathcal{I}_\mathcal{N}$ and $h \in \mathcal{H}$, and
  * $h_i P_i^t h$ for any $i \in \{f(1), f(2), \ldots, f(t-1)\} \cap \mathcal{I}_\mathcal{E}$ and $h \in (\mathcal{H} \setminus \{h_i\}) \cup \{h_0\}$.

  By *individual rationality* of ϕ, $\phi[R^1](i) = h_i$ for all $i \in \{f(1), f(2), \ldots, f(t-1)\} \cap \mathcal{I}_\mathcal{E}$, and $\phi[R^1](i) = h_0$ for all $i \in \{f(1), f(2), \ldots, f(t-1)\} \cap \mathcal{I}_\mathcal{N}$. By *Pareto efficiency* of ϕ, $\phi[R^1](i_t) = h^*$ for some $i_t \in \mathcal{I} \setminus \{f(1), f(2), \ldots, f(t-1)\}$. Let $f(t) = i_t$.

[7] The only difference in the argument in the following rounds is that, in Round t(b) for $t \in \{1, 2, \ldots, t^*\}$, the agent referred as house h_v in our argument could be either a vacant house or the occupied house of an existing tenant in $\bigcup_{s=1}^{t-1} J^s$.

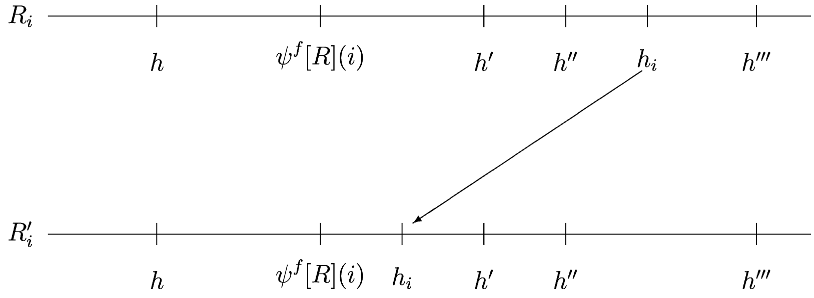

Fig. 4. Construction of Preference R'_i for Case 1 with $i \in I_E$.

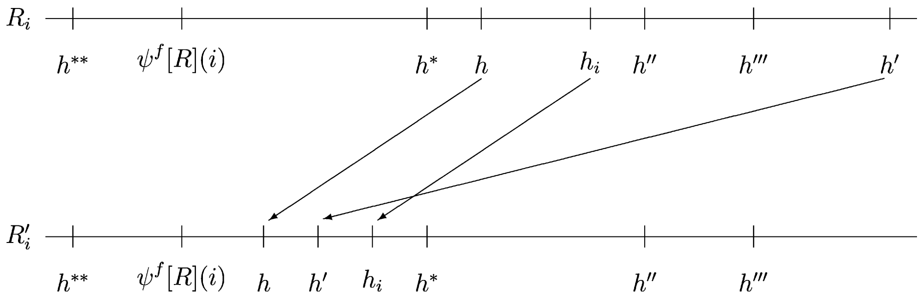

Fig. 5. Construction of Preference R'_i for Case 2 with $i \in I_E$ and $\psi^f[R](B^t)$ consists of h and h' in order of occurrence in the cycle and $\psi^f[R](i)$.

This uniquely defines an ordering $f \in \mathcal{F}$. We will prove that $\phi = \psi^f$.

Fix a problem $\langle I, H, R\rangle$. We construct matching $\psi^f[R]$ by using the YRMH-IGYT algorithm. For each Round t of the algorithm let A^t be the set of agents removed in Round t(a) of the algorithm and let B^t be the set of agents removed in Round t(b) of the algorithm.

We next construct a preference profile $R' \in \mathcal{R}(I, H)$ that will play a key role in our proof. Consider an agent $i \in I$ and let t be such that $i \in A^t \cup B^t$. Two cases are possible:

Case 1. Either $i \in A^t$ or $i \in B^t$ although she is not the highest priority agent in B^t: If $i \in I_E$ and $\psi^f[R](i) = h_i$ or if $i \in I_N$ and $\psi^f[R](i) = h_0$ then $R'_i = R_i$. Otherwise, R'_i is constructed for two subcases separately as follows:

Case 1.A. $i \in I_E$: We construct R'_i as follows:

(a) $g P'_i h \iff g P_i h$ for any $g, h \in (H \setminus \{h_i\}) \cup \{h_0\}$.

(b) $\psi^f[R](i) P'_i h_i P'_i h$ for any $h \in (H \setminus \{h_i\}) \cup \{h_0\}$ s.t. $\psi^f[R](i) P_i h$.

That is, R'_i is obtained from R_i by simply inserting house h_i right after house $\psi^f[R](i)$ and keeping the relative ranking of the rest of the houses as in R_i (see Fig. 4).

Case 1.B. $i \in I_N$: We construct R'_i as follows:

(a) $g P'_i h \iff g P_i h$ for any $g, h \in H$.

(b) $\psi^f[R](i) P'_i h_0 P'_i h$ for any $h \in H$ s.t. $\psi^f[R](i) P_i h$.

That is, R'_i is obtained from R_i by simply inserting option h_0 right after house $\psi^f[R](i)$ and keeping the relative ranking of the rest of the houses as in R_i.

Case 2. $i \in B^t$ and she is the highest priority agent in B^t under ordering f: Observe that none of the agents in B^t is assigned option h_0. Let $\psi^f[R](B^t)$ be the set of houses allocated in Round t(b) of the YRMH-IGYT algorithm.

Note that if $i \in I_E$, $\psi^f[R](i) R_i g$ for any $g \in \psi^f[R](B^t) \cup \{h_i\}$, and if $i \in I_N$, $\psi^f[R](i) R_i g$ for any $g \in \psi^f[R](B^t) \cup \{h_0\}$. Two subcases are possible:

Case 2.A. $i \in I_E$: We construct R'_i as follows:

(a) $g P'_i h \iff g P_i h$ for any $g, h \in (H \cup \{h_0\}) \setminus (\psi^f[R](B^t \setminus \{i\}) \cup \{h_i\})$.

(b) $g P'_i h \iff g P_i h$ for any $g, h \in \psi^f[R](B^t)$.

(c) $\psi^f[R](i) P'_i g P'_i h_i P'_i h$ for any $g \in \psi^f[R](B^t \setminus \{i\})$, and any $h \in (H \cup \{h_0\}) \setminus (\psi^f[R](B^t) \cup \{h_i\})$ s.t. $\psi^f[R](i) P_i h$.

That is, R'_i is obtained from R_i by inserting houses in $\psi^f[R](B^t \setminus \{i\})$ right after house $\psi^f[R](i)$ without altering their relative ranking, inserting house h_i right after that group, and keeping the relative ranking of the rest of the houses as in R_i (see Fig. 5).

Case 2.B. $i \in I_N$: We construct R'_i as follows:

(a) $g P'_i h \iff g P_i h$ for any $g, h \in H \setminus \psi^f[R](B^t \setminus \{i\})$.

(b) $gP'_i h \iff gP_i h$ for any $g, h \in \psi^f[R](B^t)$.
(c) $\psi^f[R](i)P'_i g P'_i h_0 P'_i h$ for any $g \in \psi^f[R](B^t \setminus \{i\})$, and any $h \in H \setminus \psi^f[R](B^t)$ s.t. $\psi^f[R](i)P_i h$.
That is, R'_i is obtained from R_i by inserting houses in $\psi^f[R](B^t \setminus \{i\})$ right after house $\psi^f[R](i)$ without altering their relative ranking, inserting option h_0 right after that group, and keeping the relative ranking of the rest of the houses as in R_i.

By construction, $\psi^f[R'] = \psi^f[R]$. We will prove four claims that will facilitate the proof of Proposition 2. We consider the agents in A^1 in the first two claims.

Claim 1. *For any $\hat{R}_{-A^1} \in \mathcal{R}(I \setminus A^1, H)$ and $i \in A^1$, we have $\phi[R'_{A^1}, \hat{R}_{-A^1}](i) = \psi^f[R](i)$.*

Proof of Claim 1. Fix $\hat{R}_{-A^1} \in \mathcal{R}(I \setminus A^1, H)$. By induction, we will show that $\phi[R'_{A^1}, \hat{R}_{-A^1}](i) = \psi^f[R](i)$ for all $i \in A^1$.

- Partition the agents in A^1 based on the cycle they belong to. Let $A^1_1 \subseteq A^1$ be the set of agents encountered in the first cycle in Round 1(a) of the YRMH-IGYT algorithm. Two cases are possible:
  Case 1. There is a newcomer in A^1_1: Then A^1_1 consists of a single newcomer i whose first choice is option h_0. By *individual rationality* of ϕ, we have $\phi[R'_{A^1}, \hat{R}_{-A^1}](i) = h_0 = \psi^f[R](i)$.
  Case 2. There is no newcomer in A^1_1: By *individual rationality* we have $\phi[R'_{A^1}, \hat{R}_{-A^1}](i) \in \{\psi^f[R](i), h_i\}$ for any $i \in A^1_1$. Moreover $\psi^f[R](i)R'_i h_i$ for any $i \in A^1_1$. Also we have $\psi^f[R](A^1_1) = \bigcup_{j \in A^1_1}\{h_j\}$. Hence by *Pareto efficiency*, $\phi[R'_{A^1}, \hat{R}_{-A^1}](i) = \psi^f[R](i)$ for any $i \in A^1_1$.
- Let $A^1_t \subseteq A^1$ be the set of agents removed in tth cycle in Round 1(a) of the YRMH-IGYT algorithm. In the inductive step, assume that for any agent j removed in the previous cycles, $\phi[R'_{A^1}, \hat{R}_{-A^1}](j) = \psi^f[R](j)$. Two cases are possible:
  Case 1. There is a newcomer in A^1_t: Then A^1_t consists of a single newcomer i whose first choice is option h_0 among $(H \setminus \bigcup_{t^*<t} \psi^f[R](A^1_{t^*})) \cup \{h_0\}$. By the inductive assumption since $\bigcup_{t^*<t} \phi[R](A^1_{t^*}) = \cup_{t^*<t} \psi^f[R](A^1_{t^*})$, then by *individual rationality* of ϕ, we have $\phi[R'_{A^1}, \hat{R}_{-A^1}](i) = h_0 = \psi^f[R](i)$.
  Case 2. There is no newcomer in A^1_t: By the inductive assumption, since $\bigcup_{t^*<t} \phi[R](A^1_{t^*}) = \bigcup_{t^*<t} \psi^f[R](A^1_{t^*})$, then by *individual rationality* of ϕ, we have $\phi[R'_{A^1}, \hat{R}_{-A^1}](i) \in \{\psi^f[R](i), h_i\}$ for any $i \in A^1_t$. Moreover $\psi^f[R](i)R'_i h_i$ for any $i \in A^1_t$, and $\psi^f[R](A^1_t) = \bigcup_{j \in A^1_t} h_j$. Hence by *Pareto efficiency*, $\phi[R'_{A^1}, \hat{R}_{-A^1}](i) = \psi^f[R](i)$ for any $i \in A^1_t$. □

Note that the proof of Claim 1 is entirely driven by *Pareto efficiency* and *individual rationality* of ϕ. Therefore, it directly implies the following corollary.

Corollary 1. *For any $\hat{R}_{-A^1} \in \mathcal{R}(I \setminus A^1, H)$, any* Pareto-efficient *and* individually rational *matching μ for problem $(R'_{A^1}, \hat{R}_{-A^1})$, and any $i \in A^1$, we have $\mu(i) = \psi^f[R](i)$.*

Claim 2. *For any $\hat{R}_{-A^1} \in \mathcal{R}(I \setminus A^1, H)$, and any $i \in A^1$, we have $\phi[R_{A^1}, \hat{R}_{-A^1}](i) = \psi^f[R](i)$.*

Proof of Claim 2. Fix $\hat{R}_{-A^1} \in \mathcal{R}(I \setminus A^1, H)$. For any $J \subseteq A^1$, we will prove that $\phi[R_J, R'_{A^1 \setminus J}, \hat{R}_{-A^1}](i) = \psi^f[R](i)$ for all $i \in A^1$ by induction on the size of J.

- Let $J = \{j\} \subseteq A^1$. If $\psi^f[R](j) = \begin{cases} h_j & \text{if } j \in I_E \\ h_0 & \text{if } j \in I_N \end{cases}$ then $R'_j = R_j$ by construction of R'_j. In this case, by Claim 1

$$\phi\big[R_j, R'_{A^1\setminus\{j\}}, \hat{R}_{-A^1}\big](j) = \phi\big[R'_{A^1}, \hat{R}_{-A^1}\big](j) = \psi^f[R](j).$$

Suppose $\psi^f[R](j) \neq h_j$ if $j \in I_E$ and $\psi^f[R](j) \neq h_0$ if $j \in I_N$. By *strategy-proofness* of ϕ,

$$\phi\big[R_j, R'_{A^1\setminus\{j\}}, \hat{R}_{-A^1}\big](j) R_j \phi\big[R'_{A^1}, \hat{R}_{-A^1}\big](j) \quad \text{and} \quad \phi\big[R'_{A^1}, \hat{R}_{-A^1}\big](j) R'_j \phi\big[R_j, R'_{A^1\setminus\{j\}}, \hat{R}_{-A^1}\big](j).$$

The above relation, construction of R'_j, and Claim 1 imply that (see, for example, Fig. 6 for the case $j \in I_E$)

$$\phi\big[R_j, R'_{A^1\setminus\{j\}}, \hat{R}_{-A^1}\big](j) = \phi\big[R'_{A^1}, \hat{R}_{-A^1}\big](j) = \psi^f[R](j).$$

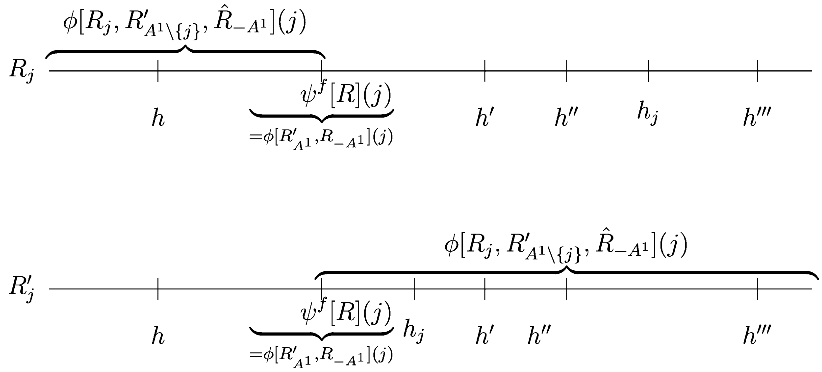

Fig. 6. For the case with $j \in I_E$, $\phi[R_j, R'_{A^1\setminus\{j\}}, \hat{R}_{-A^1}](j) = \phi[R'_{A^1}, \hat{R}_{-A^1}](j) = \psi^f[R](j)$ by *strategy-proofness*.

Therefore, while problems $(R_j, R'_{A^1\setminus\{j\}}, \hat{R}_{-A^1})$ and $(R'_{A^1}, \hat{R}_{-A^1})$ differ in preferences of agent j, her assignment under ϕ does not differ in these two problems. Hence matching $\phi[R_j, R'_{A^1\setminus\{j\}}, \hat{R}_{-A^1}]$ not only has to be *Pareto-efficient* and *individually rational* under $(R_j, R'_{A^1\setminus\{j\}}, \hat{R}_{-A^1})$ but also under $(R'_{A^1}, \hat{R}_{-A^1})$, and therefore, by Corollary 1,

$$\phi\big[R_j, R'_{A^1\setminus\{j\}}, \hat{R}_{-A^1}\big](i) = \psi^f[R](i) \quad \text{for all } i \in A^1.$$

- Fix $k \in \{1, \ldots, |A^1| - 1\}$. In the inductive step, assume that for any $J \subset A^1$ with $|J| \leqslant k$,

$$\phi\big[R_J, R'_{A^1\setminus J}, \hat{R}_{-A^1}\big](i) = \psi^f[R](i) \quad \text{for all } i \in A^1. \tag{1}$$

Fix $J \subseteq A^1$ such that $|J| = k + 1$. Fix $j \in J$. If $\psi^f[R](j) = \begin{cases} h_j & \text{if } j \in I_E \\ h_0 & \text{if } j \in I_N \end{cases}$ then $R'_j = R_j$ by construction of R'_j. In this case,

$$\phi\big[R_J, R'_{A^1\setminus J}, \hat{R}_{-A^1}\big](j) = \phi\big[R_{J\setminus\{j\}}, R'_{A^1\setminus(J\setminus\{j\})}, \hat{R}_{-A^1}\big](j) = \psi^f[R](j), \tag{2}$$

where the second equality follows from the inductive assumption Eq. (1) (since $|J \setminus \{j\}| = k$).
Suppose $\psi^f[R](j) \neq h_j$. By *strategy-proofness* of ϕ, we have

$$\phi\big[R_J, R'_{A^1\setminus J}, \hat{R}_{-A^1}\big](j) R_j \phi\big[R_{J\setminus\{j\}}, R'_{A^1\setminus(J\setminus\{j\})}, \hat{R}_{-A^1}\big](j) \quad \text{and}$$
$$\phi\big[R_{J\setminus\{j\}}, R'_{A^1\setminus(J\setminus\{j\})}, \hat{R}_{-A^1}\big](j) R'_j \phi\big[R_J, R'_{A^1\setminus J}, \hat{R}_{-A^1}\big](j).$$

The above relation and the construction of R'_j imply that

$$\phi\big[R_J, R'_{A^1\setminus J}, \hat{R}_{-A^1}\big](j) = \phi\big[R_{J\setminus\{j\}}, R'_{A^1\setminus(J\setminus\{j\})}, \hat{R}_{-A^1}\big](j) = \psi^f[R](j), \tag{3}$$

where the second equality follows from the inductive assumption Eq. (1). Since the choice of $j \in J$ is arbitrary, Eq. (2) or Eq. (3) hold for any $j \in J$. Therefore while problems $(R_J, R'_{A^1\setminus J}, \hat{R}_{-A^1})$ and $(R'_{A^1}, \hat{R}_{-A^1})$ differ in preferences of agents in J, their assignments under ϕ do not differ in these two problems. Hence matching $\phi[R_J, R'_{A^1\setminus J}, \hat{R}_{-A^1}]$ not only has to be *Pareto-efficient* and *individually rational* under $(R_J, R'_{A^1\setminus J}, \hat{R}_{-A^1})$ but also under $(R'_{A^1}, \hat{R}_{-A^1})$, and therefore, by Corollary 1,

$$\phi\big[R_J, R'_{A^1\setminus J}, \hat{R}_{-A^1}\big](i) = \psi^f[R](i) \quad \text{for all } i \in A^1,$$

completing the induction and the proof of Claim 2. □

Let $B^1 = \{i_1, \ldots, i_k\}$ and let $(h_v, i_1, h_{i_2}, i_2, \ldots, h_{i_k}, i_k)$ be the cycle removed in Round 1(b) of the YRMH-IGYT algorithm where agent i_1 is the highest priority agent in $I \setminus A^1$ under ordering f, and house h_v is a vacant house. In order to simplify the notation, let

$$h_{i_{k+1}} \equiv h_v.$$

We have

$$\psi^f[R](i_\ell) = h_{i_{\ell+1}} \quad \text{for all } \ell \in \{1, \ldots, k\}.$$

We consider the agents in B^1 in the next two claims.

Claim 3. *$\phi[R'_{B^1}, R_{-B^1}](i) = \psi^f[R](i)$ for all $i \in B^1$.*

Proof of Claim 3. First of all, observe that $\phi[R'_{B^1}, R_{-B^1}](i) = \psi^f[R](i)$ for all $i \in A^1$ by Claim 2. We will prove the claim by contradiction. Suppose that there exists an agent $i_\ell \in B^1$ such that $\phi[R'_{B^1}, R_{-B^1}](i_\ell) \neq \psi^f[R](i_\ell) = \psi^f[R'](i_\ell) = h_{i_{\ell+1}}$. Pick the last such agent in the cycle. Then

$$\phi\big[R'_{B^1}, R_{-B^1}\big](i_m) = h_{i_{m+1}} \quad \text{for all } m \in \{\ell+1, \ldots, k\} \quad \text{by the choice of } \ell,$$

$$\phi\big[R'_{B^1}, R_{-B^1}\big](i_\ell) = h_{i_\ell} \quad \text{by Claim 2 and } \textit{individual rationality} \text{ of } \phi,$$

$$\phi\big[R'_{B^1}, R_{-B^1}\big](i_{\ell-1}) = h_{i_{\ell-1}} \quad \text{by above relation, Claim 2 and } \textit{individual rationality} \text{ of } \phi,$$

$$\vdots$$

$$\phi\big[R'_{B^1}, R_{-B^1}\big](i_2) = h_{i_2} \quad \text{by above relation, Claim 2 and } \textit{individual rationality} \text{ of } \phi.$$

Since i_1 is the highest priority agent in $I \setminus A^1$, Case 2 applies in construction of R'_{i_1}. If $i_1 \in I_E$, by Claim 2 and *individual rationality* of ϕ, we have $\phi[R'_{B^1}, R_{-B^1}](i_1) \in \{h_{i_1}, \ldots, h_{i_{k+1}}\}$, and since all but houses h_{i_1} and $h_{i_{\ell+1}}$ are assigned to other agents by above relations,

$$\phi\big[R'_{B^1}, R_{-B^1}\big](i_1) \in \{h_{i_1}, h_{i_{\ell+1}}\}.$$

On the other hand, if $i_1 \in I_N$, by *individual rationality* of ϕ we have $\phi[R'_{B^1}, R_{-B^1}](i_1) \in \{h_0, h_{i_2}, \ldots, h_{i_{k+1}}\}$, and since all but house $h_{i_{\ell+1}}$ is assigned to other agents by above relations,

$$\phi\big[R'_{B^1}, R_{-B^1}\big](i_1) \in \{h_0, h_{i_{\ell+1}}\}.$$

But house $h_{i_{\ell+1}}$ can neither be left unmatched nor be matched with agent i_1 under $\phi[R'_{B^1}, R_{-B^1}]$ for otherwise assigning house $h_{i_{m+1}}$ to agent i_m for all $m \in \{1, \ldots, \ell\}$ (and keeping the other assignments the same) would result in a Pareto improvement under (R'_{B^1}, R_{-B^1}). Therefore,

$$\phi\big[R'_{B^1}, R_{-B^1}\big](i_1) = \begin{cases} h_{i_1} & \text{if } i_1 \in I_E \\ h_0 & \text{if } i_1 \in I_N \end{cases} \quad \text{and} \quad \phi\big[R'_{B^1}, R_{-B^1}\big](j_1) = h_{i_{\ell+1}} \quad \text{for some } j_1 \in I \setminus \big(A^1 \cup B^1\big).$$

We will iteratively construct a set of agents S and a restricted preference profile R''_S. Set S and profile R''_S will be used to reduce the problem by removing agents in $T = I \setminus (\{i_1\} \cup S)$ and their assigned houses $\phi[R'_{B^1}, R''_S, R_{-B^1 \cup S}](T)$. The reduced problem will be well-defined by the construction of S and R''_S. By invoking *consistency* in the reduced problem, we will be able to show the required contradiction.

Iteratively form set S as follows:

Step 1. Let $j_1 \in S$ (i.e., agent j_1 is the first agent to be included in set S). Recall that $\phi[R'_{B^1}, R_{-B^1}](j_1) = h_{i_{\ell+1}}$. Let preferences $R''_{j_1} \in \mathcal{R}(j_1, H)$ be such that

$$R''_{j_1}: \begin{cases} \underbrace{\phi[R'_{B^1}, R_{-B^1}](j_1)}_{=h_{i_{\ell+1}}} P''_{j_1} h_{j_1} P''_{j_1} h & \text{for all } h \in (H \setminus \{h_{i_{\ell+1}}, h_{j_1}\}) \cup \{h_0\} & \text{if } j_1 \in I_E, \\ \underbrace{\phi[R'_{B^1}, R_{-B^1}](j_1)}_{=h_{i_{\ell+1}}} P''_{j_1} h_0 P''_{j_1} h & \text{for all } h \in H \setminus \{h_{i_{\ell+1}}\} & \text{if } j_1 \in I_N. \end{cases}$$

Consider the problem $(R'_{B^1}, R''_{j_1}, R_{-B^1 \cup \{j_1\}})$. By Claim 2,

$$\phi\big[R'_{B^1}, R''_{j_1}, R_{-B^1 \cup \{j_1\}}\big](i) = \psi^f[R](i) \quad \text{for all } i \in A^1. \tag{4}$$

By *strategy-proofness* of ϕ,

$$\phi\big[R'_{B^1}, R''_{j_1}, R_{-B^1 \cup \{j_1\}}\big](j_1) R''_{j_1} \underbrace{\phi\big[R'_{B^1}, R_{-B^1}\big](j_1)}_{=h_{i_{\ell+1}}}$$

and since house $h_{i_{\ell+1}}$ is the top choice under R''_{j_1}

$$\phi\big[R'_{B^1}, R''_{j_1}, R_{-B^1 \cup \{j_1\}}\big](j_1) = \phi\big[R'_{B^1}, R_{-B^1}\big](j_1) = h_{i_{\ell+1}}. \tag{5}$$

Therefore

$$\phi\big[R'_{B^1}, R''_{j_1}, R_{-B^1\cup\{j_1\}}\big](i_{\ell+1}) = h_{i_{\ell+2}} \quad \text{by Eqs. (4), (5), and } \textit{individual rationality} \text{ of } \phi,$$

$$\vdots$$

$$\phi\big[R'_{B^1}, R''_{j_1}, R_{-B^1\cup\{j_1\}}\big](i_k) = h_{i_{k+1}} \quad \text{by Eq. (4), above relation, and } \textit{individual rationality} \text{ of } \phi$$

and

$$\phi\big[R'_{B^1}, R''_{j_1}, R_{-B^1\cup\{j_1\}}\big](i_\ell) = h_{i_\ell} \quad \text{by Eqs. (4), (5), and } \textit{individual rationality} \text{ of } \phi$$

$$\vdots$$

$$\phi\big[R'_{B^1}, R''_{j_1}, R_{-B^1\cup\{j_1\}}\big](i_2) = h_{i_2} \quad \text{by Eq. (4), above relation, and } \textit{individual rationality} \text{ of } \phi.$$

Eq. (4), above relations, and *individual rationality* of ϕ imply

$$\phi\big[R'_{B^1}, R''_{j_1}, R_{-B^1\cup\{j_1\}}\big](i_1) = \begin{cases} h_{i_1} & \text{if } i_1 \in I_E, \\ h_0 & \text{if } i_1 \in I_N. \end{cases}$$

Step 2. If $j_1 \in I_N$ or $j_1 \in I_E$ and there is no agent $j_2 \in I \setminus (A^1 \cup B^1)$ such that $\phi[R'_{B^1}, R''_{j_1}, R_{-B^1\cup\{j_1\}}](j_2) = h_{j_1}$ then terminate the construction of S. (Thus $S = \{j_1\}$.) Otherwise such agent j_2 is the second agent to include in set S and let preferences $R''_{j_2} \in \mathcal{R}(j_2, H)$ be such that

$$R''_{j_2}: \quad \begin{cases} \underbrace{\phi[R'_{B^1}, R''_{j_1}, R_{-B^1\cup\{j_1\}}](j_2)}_{=h_{j_1}} P''_{j_2} h_{j_2} P''_{j_2} h \quad \text{for all } h \in (H \setminus \{h_{j_1}, h_{j_2}\}) \cup \{h_0\} & \text{if } j_2 \in I_E, \\ \underbrace{\phi[R'_{B^1}, R''_{j_1}, R_{-B^1\cup\{j_1\}}](j_2)}_{=h_{j_1}} P''_{j_2} h_0 P''_{j_2} h \quad \text{for all } h \in H \setminus \{h_{j_1}\} & \text{if } j_2 \in I_N. \end{cases}$$

By Claim 2,

$$\phi\big[R'_{B^1}, R''_{\{j_1,j_2\}}, R_{-B^1\cup\{j_1,j_2\}}\big](i) = \psi^f[R](i) \quad \text{for all } i \in A^1. \tag{6}$$

By *strategy-proofness* of ϕ,

$$\phi\big[R'_{B^1}, R''_{\{j_1,j_2\}}, R_{-B^1\cup\{j_1,j_2\}}\big](j_2) R''_{j_2} \underbrace{\phi\big[R'_{B^1}, R''_{j_1}, R_{-B^1\cup\{j_1\}}\big](j_2)}_{=h_{j_1}}$$

which in turn implies

$$\phi\big[R'_{B^1}, R''_{\{j_1,j_2\}}, R_{-B^1\cup\{j_1,j_2\}}\big](j_2) = \phi\big[R'_{B^1}, R''_{j_1}, R_{-B^1\cup\{j_1\}}\big](j_2) = h_{j_1}.$$

Therefore by *individual rationality* of ϕ, Eq. (6), and construction of R''_{j_2},

$$\phi\big[R'_{B^1}, R''_{\{j_1,j_2\}}, R_{-B^1\cup\{j_1,j_2\}}\big](j_1) = h_{i_{\ell+1}},$$

which in turn implies (using a similar argument as in Step 1)

$$\phi\big[R'_{B^1}, R''_{\{j_1,j_2\}}, R_{-B^1\cup\{j_1,j_2\}}\big](i_m) = h_{i_{m+1}} \quad \text{for all } m \in \{\ell+1, \dots, k\},$$

$$\phi\big[R'_{B^1}, R''_{\{j_1,j_2\}}, R_{-B^1\cup\{j_1,j_2\}}\big](i_m) = h_{i_m} \quad \text{for all } m \in \{2, \dots, \ell\},$$

$$\phi\big[R'_{B^1}, R''_{\{j_1,j_2\}}, R_{-B^1\cup\{j_1,j_2\}}\big](i_1) = \begin{cases} h_{i_1} & \text{if } i_1 \in I_E, \\ h_0 & \text{if } i_1 \in I_N. \end{cases}$$

We continue iteratively and form set $S = \{j_1, j_2, \dots, j_s\} \subseteq I \setminus (A^1 \cup B^1)$ and preference profile $(R'_{B^1}, R''_S, R_{-B^1\cup S})$ such that

$$\text{if } j_s \in I_E \quad \text{then } \phi\big[R'_{B^1}, R''_S, R_{-B^1\cup S}\big](i) \neq h_{j_s} \quad \text{for any } i \in I,$$

$$\phi\big[R'_{B^1}, R''_S, R_{-B^1\cup S}\big](j_m) = h_{j_{m-1}} \quad \text{for all } m \in \{2, \dots, s\},$$

$$\phi\big[R'_{B^1}, R''_S, R_{-B^1\cup S}\big](j_1) = h_{i_{\ell+1}}, \quad \text{and}$$

$$\phi\big[R'_{B^1}, R''_S, R_{-B^1\cup S}\big](i_1) = \begin{cases} h_{i_1} & \text{if } i_1 \in I_E, \\ h_0 & \text{if } i_1 \in I_N. \end{cases}$$

Observe that $\{j_1, \ldots, j_{s-1}\} \subseteq I_E$, and one of the following statements holds: (i) $j_s \in I_N$, or (ii) $j_s \in I_E$ and there is no agent $i \in I$ such that $\phi[R'_{B^1}, R''_S, R_{-B^1 \cup S}](i) = h_{j_s}$. For otherwise, agent i would also be included in set S. Therefore, upon removing agents in $T = I \setminus (\{i_1\} \cup S)$ and their assigned houses $G = \phi[R'_{B^1}, R''_S, R_{-B^1 \cup S}](T)$, the reduced problem $(R'^{-G}_{i_1}, R''^{-G}_S)$ is well-defined. Note that the set of remaining agents is $\{i_1\} \cup S$, and house $h_{i_{\ell+1}}$ is a vacant house in the reduced problem (possibly together with other vacant houses). By *consistency* of ϕ, we have

$$\phi\big[R'^{-G}_{i_1}, R''^{-G}_S\big](i) = \phi\big[R'_{B^1}, R''_S, R_{-B^1 \cup S}\big](i) \quad \text{for all } i \in \{i_1\} \cup S.$$

In the rest of the proof, for the sake of notation we set

$$h_{j_0} \equiv h_{i_{\ell+1}}.$$

Note that the preference relation profile $(R'^{-G}_{i_1}, R''^{-G}_S) \in \mathcal{R}(\{i_1\} \cup G, H \setminus G)$ is given as follows:

$$R'^{-G}_{i_1}: \quad \begin{cases} h_{j_0} P'^{-G}_{i_1} h_{i_1} P'^{-G}_{i_1} \cdots & \text{if } i_1 \in I_E, \\ h_{j_0} P'^{-G}_{i_1} h_0 P'^{-G}_{i_1} \cdots & \text{if } i_1 \in I_N, \end{cases}$$

$$R''^{-G}_{S \setminus \{j_s\}}: \quad \begin{matrix} h_{j_0} P''^{-G}_{j_1} h_{j_1} P''^{-G}_{j_1} \cdots \\ h_{j_1} P''^{-G}_{j_2} h_{j_2} P''^{-G}_{j_2} \cdots \\ \vdots \\ h_{j_{s-2}} P''^{-G}_{j_{s-1}} h_{j_{s-1}} P''^{-G}_{j_{s-1}} \cdots \end{matrix}$$

$$R''^{-G}_{j_s}: \quad \begin{cases} h_{j_{s-1}} P''^{-G}_{j_s} h_{j_s} P''^{-G}_{j_s} \cdots & \text{if } j_s \in I_E, \\ h_{j_{s-1}} P''^{-G}_{j_s} h_0 P''^{-G}_{j_s} \cdots & \text{if } j_s \in I_N. \end{cases}$$

We consider the remaining houses $H \setminus G$ and construct the following preference relation $R''^{-G}_{i_1} \in \mathcal{R}(i_1, H \setminus G)$:

$$R''^{-G}_{i_1}: \quad \begin{cases} h_{j_0} P''^{-G}_{i_1} h_{j_1} P''^{-G}_{i_1} \cdots P''^{-G}_{i_1} h_{j_{s-1}} P''^{-G}_{i_1} h_{i_1} P''^{-G}_{i_1} h & \\ \quad \text{for all } h \in (H \setminus (G \cup \{h_{j_0}, \ldots, h_{j_{s-1}}, h_{i_1}\})) \cup \{h_0\} & \text{if } i_1 \in I_E, \\ h_{j_0} P''^{-G}_{i_1} h_{j_1} P''^{-G}_{i_1} \cdots P''^{-G}_{i_1} h_{j_{s-1}} P''^{-G}_{i_1} h_0 P''^{-G}_{i_1} h & \\ \quad \text{for all } h \in H \setminus (G \cup \{h_{j_0}, \ldots, h_{j_{s-1}}\}) & \text{if } i_1 \in I_N. \end{cases}$$

By *strategy-proofness* of ϕ,

$$\underbrace{\phi\big[R'^{-G}_{i_1}, R''^{-G}_S\big](i_1)}_{= \begin{cases} h_{i_1} & \text{if } i_1 \in I_E, \\ h_0 & \text{if } i_1 \in I_N \end{cases}} R'^{-G}_{i_1} \phi\big[R''^{-G}_{\{i_1\} \cup S}\big](i_1).$$

We have $h_{j_0} \in \psi^f[R](B^1)$, we have $h_{j_0} P'^{-G}_{i_1} h_{i_1}$ and therefore $\phi[R''^{-G}_{\{i_1\} \cup S}](i_1) \neq h_{j_0}$. Thus,

$$\phi\big[R''^{-G}_{\{i_1\} \cup S}\big](i_1) \in \begin{cases} \{h_{j_1}, \ldots, h_{j_{s-1}}, h_{i_1}\} & \text{if } i_1 \in I_E, \\ \{h_{j_1}, \ldots, h_{j_{s-1}}, h_0\} & \text{if } i_1 \in I_N. \end{cases}$$

Two cases are possible:

Case 1. $\phi[R''^{-G}_{\{i_1\} \cup S}](i_1) \in \{h_{j_1}, \ldots, h_{j_{s-1}}\}$:

Let $\phi[R''^{-G}_{\{i_1\} \cup S}](i_1) = h_{j_m}$ such that $m < s$. Thus $j_m \in I_E$. Then by *individual rationality*

$$\phi\big[R''^{-G}_{\{i_1\} \cup S}\big](j_p) = \begin{cases} h_{j_p} & \text{if } j_p \in I_E \\ h_0 & \text{if } j_p \in I_N \end{cases} \quad \text{for all } p \in \{m+1, \ldots, s\},$$

$$\phi\big[R''^{-G}_{\{i_1\} \cup S}\big](j_p) = h_{j_{p-1}} \quad \text{for all } p \in \{1, \ldots, m\}.$$

Therefore, upon removing all agents except $\{i_1, j_m\}$ and all houses except

$$G' = \begin{cases} \{h_{i_1}, h_{j_{m-1}}, h_{j_m}\} & \text{if } i_1 \in I_E, \\ \{h_{j_{m-1}}, h_{j_m}\} & \text{if } i_1 \in I_N \end{cases}$$

from the reduced problem $R''^{-G}_{\{i_1\} \cup S}$, the further reduced problem $R''^{G'}_{\{i_1, j_m\}}$ is well-defined. That is because, under $\phi[R''^{-G}_{\{i_1\} \cup S}]$, h_{i_1} is unassigned in case $i_1 \in I_E$, h_{j_m} is assigned to agent i_1, and $h_{j_{m-1}}$ is assigned to agent j_m. In this further reduced problem, house $h_{j_{m-1}}$ is the unique vacant house. By *consistency* of ϕ, we have

$$\phi\big[R''^{G'}_{\{i_1,j_m\}}\big](i_1)=\phi\big[R''^{-G}_{\{i_1\}\cup S}\big](i_1)=h_{j_m},$$
$$\phi\big[R''^{G'}_{\{i_1,j_m\}}\big](j_m)=\phi\big[R''^{-G}_{\{i_1\}\cup S}\big](j_m)=h_{j_{m-1}}.$$

Note that the preference profile $R''^{G'}_{\{i_1,j_m\}}\in\mathcal{R}(\{i_1,j_m\},G')$ is given as follows:

$$R''^{G'}_{i_1}:\quad\begin{cases}h_{j_{m-1}}P''^{G'}_{i_1}h_{j_m}P''^{G'}_{i_1}h_{i_1}P''^{G'}_{i_1}h_0 & \text{if } i_1\in I_E,\\ h_{j_{m-1}}P''^{G'}_{i_1}h_{j_m}P''^{G'}_{i_1}h_0 & \text{if } i_1\in I_N,\end{cases}\qquad R''^{G'}_{j_m}:\quad h_{j_{m-1}}P''^{G'}_{j_m}h_{j_m}P''^{G'}_{j_m}\cdots.$$

We consider the houses in G' and construct the following preference relation $\tilde{R}^{G'}_{i_1}\in\mathcal{R}(i_1,G')$:

$$\tilde{R}^{G'}_{i_1}:\quad\begin{cases}h_{j_{m-1}}\tilde{P}^{G'}_{i_1}h_{i_1}\tilde{P}^{G'}_{i_1}\cdots & \text{if } i_1\in I_E,\\ h_{j_{m-1}}\tilde{P}^{G'}_{i_1}h_0\tilde{P}^{G'}_{i_1}\cdots & \text{if } i_1\in I_N.\end{cases}$$

By *strategy-proofness* of ϕ,

$$\underbrace{\phi\big[R''^{G'}_{\{i_1,j_m\}}\big](i_1)}_{=h_{j_m}}R''^{G'}_{i_1}\phi\big[\tilde{R}^{G'}_{i_1},R''^{G'}_{j_m}\big](i_1).$$

Since $h_{j_{m-1}}P''^{G'}_{i_1}h_{j_m}$ by construction, $\phi[\tilde{R}^{G'}_{i_1},R''^{G'}_{j_m}](i_1)\neq h_{j_{m-1}}$. Therefore by *individual rationality* of ϕ,

$$\phi\big[\tilde{R}^{G'}_{i_1},R''^{G'}_{j_m}\big](i_1)=\begin{cases}h_{i_1} & \text{if } i_1\in I_E,\\ h_0 & \text{if } i_1\in I_N,\end{cases}$$

and this together with *Pareto-efficiency* of ϕ imply

$$\phi\big[\tilde{R}^{G'}_{i_1},R''^{G'}_{j_m}\big](j_m)=h_{j_{m-1}}.$$

Recall that i_1 is the highest priority agent in $I\setminus A^1$ under ordering f. In particular, i_1 has higher priority than j_m, since $j_m\in I\setminus(A^1\cup B^1)$. Let $i_1=f(t)$ for some t. Consider the profile R^t used in construction of f. Any agent i ordered before i_1 has h_i as her first choice under R^t if $i\in\mathcal{I}_\mathcal{E}$ and has h_0 as her first choice under R^t if $i\in\mathcal{I}_\mathcal{N}$, whereas any other agent i has the vacant house h^* as her first choice, h_i as her second choice under R^t if $i\in\mathcal{I}_\mathcal{E}$, and h_0 as her second choice under R^t if $i\in\mathcal{I}_\mathcal{N}$. We have $\phi[R^t](i_1)=h^*$ and $\phi[R^t](i)=\begin{cases}h_i & \text{if } i\in\mathcal{I}_\mathcal{E}\\ h_0 & \text{if } i\in\mathcal{I}_\mathcal{N}\end{cases}$ for all $i\in I\setminus\{i_1\}$ by construction of f and *individual rationality* of ϕ. Therefore, upon removing all agents except $\{i_1,j_m\}$ and all houses except

$$G^*=\begin{cases}\{h_{i_1},h_{j_m},h^*\} & \text{if } i_1\in\mathcal{I}_\mathcal{E},\\ \{h_{j_m},h^*\} & \text{if } i_1\in\mathcal{I}_\mathcal{N}\end{cases}$$

from problem R^t, the reduced problem $R^{tG^*}_{\{i_1,j_m\}}$ is well-defined. (That is because, h_{i_1} is unmatched if $i_1\in\mathcal{I}_\mathcal{E}$, h_{j_m} is matched to j_m, and h^* is matched to i_1 under $\phi[R^t]$.) By *consistency* of ϕ,

$$\phi\big[R^{tG^*}_{\{i_1,j_m\}}\big](i_1)=\phi\big[R^t\big](i_1)=h^*\quad\text{and}\quad\phi\big[R^{tG^*}_{\{i_1,j_m\}}\big](j_m)=\phi\big[R^t\big](j_m)=h_{j_m}.$$

Note that the preference profile $R^{tG^*}_{\{i_1,j_m\}}\in\mathcal{R}\{i_1,j_m\},G^*$ is given as follows:

$$R^{tG^*}_{i_1}:\quad\begin{cases}h^*P^{tG^*}_{i_1}h_{i_1}P^{tG^*}_{i_1}\cdots & \text{if } i_1\in\mathcal{I}_\mathcal{E},\\ h^*P^{tG^*}_{i_1}h_0P^{tG^*}_{i_1}\cdots & \text{if } i_1\in\mathcal{I}_\mathcal{N},\end{cases}\qquad R^{tG^*}_{j_m}:\quad h^*P^{tG^*}_{j_m}h_{j_m}P^{tG^*}_{j_m}\cdots.$$

There is a single vacant house in both reduced problems $(\tilde{R}^{G'}_{i_1},R''^{G'}_{j_m})$ and $R^{tG^*}_{\{i_1,j_m\}}$ while the agents and the occupied houses are the same. Under the profile $(\tilde{R}^{G'}_{i_1},R''^{G'}_{j_m})$ each agent ranks the vacant house $h_{j_{m-1}}$ as the first choice, her occupied house if she is an existing tenant or option h_0 if she is a newcomer as the second choice. Similarly under profile $R^{tG^*}_{\{i_1,j_m\}}$ each agent ranks the vacant house h^* as the first choice, her occupied house if she is an existing tenant or option h_0 if she is a newcomer as second choice.[8] However, agent j_m is assigned the top ranked vacant

[8] Observe that for each agent $i\in\{i_1,j_m\}$ the ranking of options below $\begin{cases}h_i & \text{if } i\in\mathcal{I}_\mathcal{E}\\ h_0 & \text{if } i\in\mathcal{I}_\mathcal{N}\end{cases}$ does not matter under both preference profiles by the following reasoning: Agent i receives an option weakly preferred to $\begin{cases}h_i & \text{if } i\in\mathcal{I}_\mathcal{E}\\ h_0 & \text{if } i\in\mathcal{I}_\mathcal{N}\end{cases}$ under either profile. When she changes the ranking of houses ranked below $\begin{cases}h_i & \text{if } i\in\mathcal{I}_\mathcal{E}\\ h_0 & \text{if } i\in\mathcal{I}_\mathcal{N}\end{cases}$ by *strategy-proofness* of ϕ, she will continue to receive the same option as before. By *Pareto efficiency* of ϕ the other agent will continue to receive the same option she was receiving before.

house $h_{j_{m-1}}$ under $\phi[\tilde{R}^{G'}_{i_1}, R''^{G'}_{j_m}]$ whereas agent i_1 is assigned the top ranked vacant house h^* under $\phi[R^{tG^*}_{\{i_1,j_m\}}]$, contradicting *weak neutrality* of ϕ. Therefore, we have $\phi[R'_{B^1}, R_{-B^1}](i) = \psi^f[R](i)$ for all $i \in B^1$ completing the proof of Claim 3.

Case 2. $\phi[R''^{-G}_{\{i_1\}\cup S}](i_1) = \begin{cases} h_{i_1} & \text{if } i_1 \in I_E, \\ h_0 & \text{if } i_1 \in I_N \end{cases}$
then by Pareto efficiency of ϕ

$$\phi\big[R''^{-G}_{\{i_1\}\cup S}\big](j_p) = h_{j_{p-1}} \quad \text{for all } p \in \{1, \ldots, s\}.$$

Therefore, upon removing all agents except $\{i_1, j_s\}$ and all houses except

$$G' = \begin{cases} \{h_{i_1}, h_{j_s}, h_{j_{s-1}}\} & \text{if } i_1 \in I_E \text{ and } j_s \in I_E, \\ \{h_{i_1}, h_{j_{s-1}}\} & \text{if } i_1 \in I_N \text{ and } j_s \in I_E, \\ \{h_{j_s}, h_{j_{s-1}}\} & \text{if } i_1 \in I_N \text{ and } j_s \in I_E, \\ \{h_{j_{s-1}}\} & \text{if } i_1 \in I_N \text{ and } j_s \in I_N \end{cases}$$

from the reduced problem $R''^{-G}_{\{i_1\}\cup S}$, the further reduced problem $R''^{G'}_{\{i_1,j_s\}}$ is well-defined. (That is because, i_1 is unmatched if $i_1 \in I_N$ and h_{i_1} is matched to i_1 if $i_1 \in I_E$, $h_{j_{s-1}}$ is matched to agent j_s, and h_{j_s} is unmatched if $j_s \in I_E$ under $\phi[R''^{-G}_{\{i_1\}\cup S}]$). In this further reduced problem, house $h_{j_{s-1}}$ is the unique vacant house. By *consistency* of ϕ, we have

$$\phi\big[R''^{G'}_{\{i_1,j_s\}}\big](i_1) = \phi\big[R''^{-G}_{\{i_1\}\cup S}\big](i_1) = \begin{cases} h_{i_1} & \text{if } i_1 \in I_E, \\ h_0 & \text{if } i_1 \in I_N, \end{cases}$$

$$\phi\big[R''^{G'}_{\{i_1,j_s\}}\big](j_s) = \phi\big[R''^{-G}_{\{i_1\}\cup S}\big](j_s) = h_{j_{s-1}}.$$

Note that the preference profile $R''^{G'}_{\{i_1,j_m\}} \in \mathcal{R}(\{i_1, j_m\}, G')$ is given as follows:

$$R''^{G'}_{i_1}\colon \quad \begin{cases} h_{j_{s-1}} P''^{G'}_{i_1} h_{i_1} P''^{G'}_{i_1} \cdots & \text{if } i_1 \in I_E, \\ h_{j_{s-1}} P''^{G'}_{i_1} h_0 P''^{G'}_{i_1} \cdots & \text{if } i_1 \in I_N, \end{cases} \qquad R''^{G'}_{j_s}\colon \quad \begin{cases} h_{j_{s-1}} P''^{G'}_{j_s} h_{j_s} P''^{G'}_{j_s} \cdots & \text{if } j_s \in I_E, \\ h_{j_{s-1}} P''^{G'}_{j_s} h_0 P''^{G'}_{j_s} \cdots & \text{if } j_s \in I_N. \end{cases}$$

Recall that i_1 is the highest priority agent in $I \setminus A^1$ under ordering f. In particular, i_1 has higher priority than j_s, since $j_s \in I \setminus (A^1 \cup B^1)$. Let $i_1 = f(t)$ for some t. Consider the profile R^t used in construction of f. Any agent i ordered before i_1 has h_i as her first choice under R^t if $i \in \mathcal{I}_{\mathcal{E}}$, h_0 as her first choice under R^t if $i \in \mathcal{I}_{\mathcal{N}}$, whereas any other agent i has the vacant house h^* as her first choice and h_i as her second choice under R^t if $i \in \mathcal{I}_{\mathcal{E}}$, h_0 as her second choice under R^t if $i \in I_N$. We have $\phi[R^t](i_1) = h^*$ and $\phi[R^t](i) = \begin{cases} h_i & \text{if } i \in \mathcal{I}_{\mathcal{E}}, \\ h_0 & \text{if } i \in \mathcal{I}_{\mathcal{N}} \end{cases}$ for all $i \in I \setminus \{i_1\}$ by construction of f and *individual rationality* of ϕ. Therefore, upon removing all agents except $\{i_1, j_s\}$ and all houses except

$$G^* = \begin{cases} \{h_{i_1}, h_{j_s}, h^*\} & \text{if } i_1 \in \mathcal{I}_{\mathcal{E}} \text{ and } j_s \in \mathcal{I}_{\mathcal{E}}, \\ \{h_{i_1}, h^*\} & \text{if } i_1 \in \mathcal{I}_{\mathcal{E}} \text{ and } j_s \in \mathcal{I}_{\mathcal{N}}, \\ \{h_{j_s}, h^*\} & \text{if } i_1 \in \mathcal{I}_{\mathcal{N}} \text{ and } j_s \in \mathcal{I}_{\mathcal{E}}, \\ \{h^*\} & \text{if } i_1 \in \mathcal{I}_{\mathcal{N}} \text{ and } j_s \in \mathcal{I}_{\mathcal{N}} \end{cases}$$

from problem R^t, the reduced problem $R^{tG^*}_{\{i_1,j_s\}}$ is well-defined. (That is because, h_{i_1} is unmatched if $i_1 \in \mathcal{I}_{\mathcal{E}}$, h_{j_s} is matched to j_s if $j_s \in \mathcal{I}_{\mathcal{E}}$, and h^* is matched to i_1 under $\phi[R^t]$.) By *consistency* of ϕ,

$$\phi\big[R^{tG^*}_{\{i_1,j_s\}}\big](i_1) = \phi\big[R^t\big](i_1) = h^* \quad \text{and} \quad \phi\big[R^{tG^*}_{\{i_1,j_s\}}\big](j_s) = \phi\big[R^t\big](j_s) = \begin{cases} h_{j_s} & \text{if } j_s \in \mathcal{I}_{\mathcal{E}}, \\ h_0 & \text{if } j_s \in \mathcal{I}_{\mathcal{N}}. \end{cases}$$

Note that the preference profile $R^{tG^*}_{\{i_1,j_s\}} \in \mathcal{R}(\{i_1, j_s\}, G^*)$ is given as follows:

$$R^{tG^*}_{i_1}\colon \quad \begin{cases} h^* P^{tG^*}_{i_1} h_{i_1} P^{tG^*}_{i_1} \cdots & \text{if } i_1 \in \mathcal{I}_{\mathcal{E}}, \\ h^* P^{tG^*}_{i_1} h_0 P^{tG^*}_{i_1} \cdots & \text{if } i_1 \in \mathcal{I}_{\mathcal{N}}, \end{cases} \qquad R^{tG^*}_{j_s}\colon \quad \begin{cases} h^* P^{tG^*}_{j_s} h_{j_s} P^{tG^*}_{j_s} \cdots & \text{if } j_s \in \mathcal{I}_{\mathcal{E}}, \\ h^* P^{tG^*}_{j_s} h_0 P^{tG^*}_{j_s} \cdots & \text{if } j_s \in \mathcal{I}_{\mathcal{N}}. \end{cases}$$

There is a single vacant house in both reduced problems $R''^{G'}_{\{i_1,j_s\}}$ and $R^{tG^*}_{\{i_1,j_s\}}$ while the agents and the occupied houses are the same. Under the profile $R''^{G'}_{\{i_1,j_s\}}$ each agent ranks the vacant house $h_{j_{s-1}}$ as the first choice, her occupied house if she is an existing tenant or option h_0 if she is a newcomer as the second choice. Similarly under profile $R^{tG^*}_{\{i_1,j_s\}}$ each agent ranks the vacant house h^* as the first choice, her occupied house if she is an existing tenant or option h_0 if she is a newcomer as the second choice (see footnote 8). However, agent j_s is assigned the top ranked vacant house $h_{j_{s-1}}$ under $\phi[R''^{G'}_{\{i_1,j_s\}}]$ whereas agent i_1 is assigned the top ranked vacant house h^* under $\phi[R^{tG^*}_{\{i_1,j_m\}}]$, contradicting *weak neutrality* of ϕ. Therefore, we have $\phi[R'_{B^1}, R_{-B^1}](i) = \psi^f[R](i)$ for all $i \in B^1$ completing the proof of Claim 3. □

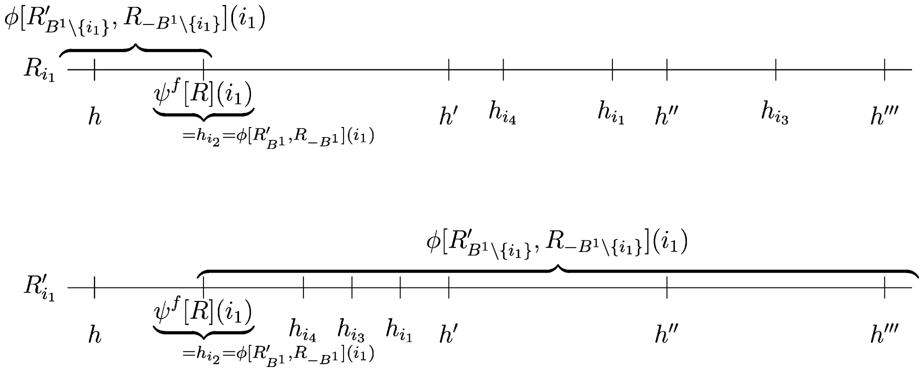

Fig. 7. When $i_1 \in I_E$, $\phi[R'_{B^1\setminus\{i_1\}}, R_{-B^1\setminus\{i_1\}}](i_1) = \phi[R'_{B^1}, R_{-B^1}](i_1) = \psi^f[R](i_1) = h_{i_2}$ by *strategy-proofness* of ϕ for the case with $B^1 = \{i_1, i_2, i_3\}$ and $h_{i_4} \equiv h_v$ is a vacant house. For the case $i_1 \in I_N$, replace h_{i_1} with h_0 in the figure.

Claim 4. *$\phi[R](i) = \psi^f[R](i)$ for all $i \in B^1$.*

Proof of Claim 4. We prove the claim by induction. Starting from preference profile (R'_{B^1}, R_{-B^1}), we will replace R'_i with R_i for each agent i in $B^1 = \{i_1, \ldots, i_k\}$ one at a time in order. Recall that $(h_v, i_1, h_{i_2}, i_2, \ldots, h_{i_k}, i_k)$ is the cycle removed in Round 1(b) of the YRMH-IGYT algorithm where agent i_1 is the highest priority agent in $I \setminus A^1$ under ordering f, and house h_v is a vacant house. Recall that $h_{i_{k+1}} \equiv h_v$. We have

$$\psi^f[R](i_\ell) = h_{i_{\ell+1}} \quad \text{for all } \ell \in \{1, \ldots, k\}.$$

- Consider the preference profile $(R'_{B^1\setminus\{i_1\}}, R_{-B^1\setminus\{i_1\}})$. By Claim 2,

$$\phi\big[R'_{B^1\setminus\{i_1\}}, R_{-B^1\setminus\{i_1\}}\big](i) = \psi^f[R](i) \quad \text{for all } i \in A^1. \tag{7}$$

By *strategy-proofness* of ϕ,

$$\phi\big[R'_{B^1\setminus\{i_1\}}, R_{-B^1\setminus\{i_1\}}\big](i_1) R_{i_1} \underbrace{\phi\big[R'_{B^1}, R_{-B^1}\big](i_1)}_{=h_{i_2}} \quad \text{and}$$

$$\underbrace{\phi\big[R'_{B^1}, R_{-B^1}\big](i_1)}_{=h_{i_2}} R'_{i_1} \phi\big[R'_{B^1\setminus\{i_1\}}, R_{-B^1\setminus\{i_1\}}\big](i_1).$$

Recall that i_1 is the highest priority agent in B^1 under ordering f. Therefore, Case 2 applies to the construction of R'_{i_1} and the above relation together with construction of R'_{i_1} imply (see Fig. 7 for the case $i_1 \in I_E$. For the case $i_1 \in I_N$, replace h_{i_1} with h_0 in the same figure).

$$\phi\big[R'_{B^1\setminus\{i_1\}}, R_{-B^1\setminus\{i_1\}}\big](i_1) = \phi\big[R'_{B^1}, R_{-B^1}\big](i_1) = \psi^f[R](i_1) = h_{i_2}, \tag{8}$$

where the second equality follows from Claim 3.
By *individual rationality* of ϕ, Eq. (7), and construction of $R'_{B^1\setminus\{i_1\}}$ (for which Case 1 applies) we have

$$\phi\big[R'_{B^1\setminus\{i_1\}}, R_{-B^1\setminus\{i_1\}}\big](i_\ell) \in \{h_{i_\ell}, h_{i_{\ell+1}}\} \quad \text{for all } \ell \in \{2, \ldots, k\}. \tag{9}$$

Then,

$$\phi\big[R'_{B^1\setminus\{i_1\}}, R_{-B^1\setminus\{i_1\}}\big](i_2) = h_{i_3} \quad \text{by Eqs. (8) and (9),}$$

$$\vdots$$

$$\phi\big[R'_{B^1\setminus\{i_1\}}, R_{-B^1\setminus\{i_1\}}\big](i_k) = h_{i_{k+1}} \quad \text{by above relation and Eq. (9).}$$

We showed that

$$\phi\big[R'_{B^1\setminus\{i_1\}}, R_{-B^1\setminus\{i_1\}}\big](i) = \phi\big[R'_{B^1}, R_{-B^1}\big](i) = \psi^f[R](i) \quad \text{for all } i \in B^1.$$

- Let $\ell \in \{2, \dots, k\}$ and $J = \{i_\ell, \dots, i_k\}$. In the inductive step, assume that

  $$\phi\big[R'_J, R_{-J}\big](i) = \psi^f[R](i) \quad \text{for all } i \in B^1.$$

  We will show that $\phi[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}](i) = \psi^f[R](i)$ for all $i \in B^1$.
  Consider preference profile $(R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}})$. By Claim 2,

  $$\phi\big[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\big](i) = \psi^f[R](i) \quad \text{for all } i \in A^1. \tag{10}$$

  By *strategy-proofness* of ϕ,

  $$\phi\big[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\big](i_\ell) R_{i_\ell} \underbrace{\phi\big[R'_J, R_{-J}\big](i_\ell)}_{=h_{i_{\ell+1}}} \quad \text{and}$$

  $$\underbrace{\phi\big[R'_J, R_{-J}\big](i_\ell)}_{=h_{i_{\ell+1}}} R'_{i_\ell} \phi\big[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\big](i_\ell)$$

  and this together with construction of R_{i_ℓ} (for which Case 1 applies) imply

  $$\phi\big[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\big](i_\ell) = \phi\big[R'_J, R_{-J}\big](i_\ell) = \psi^f[R](i_\ell) = h_{i_{\ell+1}}, \tag{11}$$

  where the second equality follows from the inductive assumption.
  By *individual rationality* of ϕ, Eq. (10), and construction of $R'_{J\setminus\{i_\ell\}}$ (for which Case 1 applies) we have

  $$\phi\big[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\big](i_m) \in \{h_{i_m}, h_{i_{m+1}}\} \quad \text{for all } m \in \{\ell+1, \dots, k\}. \tag{12}$$

  Then,

  $$\phi\big[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\big](i_{\ell+1}) = h_{i_{\ell+2}} \quad \text{by Eqs. (11) and (12),}$$
  $$\vdots$$
  $$\phi\big[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\big](i_k) = h_{i_{k+1}} \quad \text{by above relation and Eq. (12).}$$

  Hence, we showed that

  $$\phi\big[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\big](i) = \phi\big[R'_J, R_{-J}\big](i) = \psi^f[R](i) \quad \text{for all } i \in J. \tag{13}$$

  We are ready to complete the induction by invoking consistency: Upon removing agents in $J = \{i_\ell, \dots, i_k\}$ and their assignments

  $$\phi\big[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\big](J) = \phi\big[R'_J, R_{-J}\big](J) = \{h_{i_{\ell+1}}, \dots, h_{i_{k+1}}\} \tag{14}$$

  from problems $(R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}})$ and (R'_J, R_{-J}), the reduced problems are not only well-defined (recall that $h_{i_{k+1}}$ is a vacant house) but also identical. Therefore, for any $i \in I \setminus J$,

  $$\begin{aligned}\phi\big[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\big](i) &= \phi\big[R_{-J}^{-\phi[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}](J)}\big](i) \quad \text{by } \textit{consistency} \text{ of } \phi,\\ &= \phi\big[R_{-J}^{-\phi[R'_J, R_{-J}](J)}\big](i) \quad \text{by Eq. (14),}\\ &= \phi\big[R'_J, R_{-J}\big](i) \quad \text{by } \textit{consistency} \text{ of } \phi\end{aligned}$$

  and this together with Eq. (13) imply

  $$\phi\big[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\big] = \phi\big[R'_J, R_{-J}\big]. \tag{15}$$

  Eq. (15) and inductive assumption imply that

  $$\phi\big[R'_{J\setminus\{i_\ell\}}, R_{-J\setminus\{i_\ell\}}\big](i) = \psi^f[R](i) \quad \text{for all } i \in B^1,$$

  completing the induction and the proof of Claim 4. □

We are ready to complete the proof of Proposition 2. By Claim 2 and Claim 4,

$$\phi[R](i) = \psi^f[R](i) \quad \text{for all } i \in A^1 \cup B^1. \tag{16}$$

Since for any $i \in A^1 \cup B^1$, assignment $\psi[R](i)$ is either the vacant house $h_{i_{k+1}}$, the occupied house of an agent in $A^1 \cup B^1$, or option h_0, upon removing the agents in $A^1 \cup B^1$ and their assigned houses included in $\phi[R](A^1 \cup B^1) = \psi^f[R](A^1 \cup B^1)$ from the problem R, the reduced problem $R^{-\phi[R](A^1 \cup B^1)}_{-A^1 \cup B^1}$ is well-defined. For any $i \in A^2 \cup B^2$, we have

$$\begin{aligned}\phi[R](i) &= \phi\big[R^{-\phi[R](A^1 \cup B^1)}_{-A^1 \cup B^1}\big](i) \quad \text{by } \textit{consistency} \text{ of } \phi,\\ &= \psi^f\big[R^{-\phi[R](A^1 \cup B^1)}_{-A^1 \cup B^1}\big](i) \quad \text{by application of Claims 2 and 4 to } R^{-\phi[R](A^1 \cup B^1)}_{-A^1 \cup B^1} \text{ for } A^2 \cup B^2,\\ &= \psi^f\big[R^{-\psi^f[R](A^1 \cup B^1)}_{-A^1 \cup B^1}\big](i) \quad \text{by Eq. (16)},\\ &= \psi^f[R](i) \quad \text{by } \textit{consistency} \text{ of } \psi^f.\end{aligned}$$

We iteratively continue with agents in $A^3 \cup B^3$, and so on to obtain

$$\phi[R] = \psi^f[R]$$

completing the proof.

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