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Implementation in generalized matching problems

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Abstract

We search for (Nash) implementable solutions on a class of one-to-one matching problems which includes both the housing market (Shapley and Scarf, *Journal of Mathematical Economics*, 1974, 1, 23–28) and marriage problems (Gale and Shapley, *American Mathematical Monthly*, 1962, 69, 9–15). We show that the core correspondence is implementable. We show, furthermore, that any solution that is Pareto efficient, individually rational, and implementable is a supersolution of the core correspondence. That is, the core correspondence is the minimal solution that is Pareto efficient, individually rational, and implementable. A corollary of independent interest in the context of the housing market is that the core correspondence is the only single-valued solution that is Pareto efficient, individually rational, and implementable.

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1. Introduction

The main objective of the mechanism design literature is to provide 'reasonable' solutions to public decision problems. When evaluating a candidate solution, one of the questions most often asked is: Is the solution *strategy-proof*? That is:

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Do agents always have the incentive to be truthful about their preferences? Unfortunately, in most contexts it is not an easy task to find a strategy-proof solution that also satisfies some minimal normative properties. [1] As far as *matching problems* are concerned, there are both positive and negative results. Consider the *housing market* (Shapley and Scarf, 1974). In this model each agent owns one indivisible good, say a house, and has preferences over the houses held by all agents in the economy. An allocation here is a permutation of the houses among the agents. Roth and Postlewaite (1977) show that the core correspondence is single-valued, and Roth (1982a) shows that it is strategy-proof. Furthermore, Ma (1994) shows that it is the only solution that is Pareto efficient, individually rational, and strategy-proof. Another class of matching problems that has been extensively studied is the class of *marriage problems* [2] (Gale and Shapley, 1962). Here, there are two finite disjoint sets of agents interpreted as a set of men and a set of women. Each man has a preference relation over the set of women and staying single. Similarly, each woman has a preference relation over the set of men and staying single. An allocation is a matching of men and women. Gale and Shapley (1962) show that the core correspondence is well-defined, i.e. the core of each marriage problem is non-empty. Unfortunately, the results concerning strategy-proofness in marriage problems are quite discouraging. Roth (1982b) shows that there is no selection from the core correspondence that is strategy-proof. Moreover, Alcalde and Barberà (1994) show that there is no solution that is Pareto efficient, individually rational, and strategy proof.

Motivated by such different results in two apparently similar class of problems, Sönmez (1994) introduces the class of *generalized matching problems* (which include both the marriage problems and the housing market) and studies strategy-proofness in this class. He shows that there exists a solution that is Pareto efficient, individually rational, and strategy-proof only if the core correspondence is single-valued. Furthermore, if such a solution exists, it is the core correspondence itself. [3] This result has both positive and negative implications. On the positive side, it provides important non-cooperative support for the core correspondence, which a cooperative solution. Yet, it once again highlights the difficulties in obtaining strategy-proof solutions since often it is not the case that the core correspondence is single-valued. For that reason in this paper we weaken the incentive requirement and ask the following question in the context of generalized matching problems: Is it possible to construct a game form such that at equilib-

[1] Strategy-proofness was first analyzed in abstract social choice models where there are few or no restrictions on preferences. Gibbard (1973) and Satterthwaite (1975) show that, under minor conditions, strategy-proofness is equivalent to dictatorship. See Sprumont (1995) and Thomson (1994) for recent surveys of the literature on strategy-proofness.

[2] See Roth and Sotomayor (1990) for an exposition of game-theoretic modelling and analysis of marriage problems and *two-sided matching problems* in general.

[3] Sönmez (1996) obtains analogous results in the context of many-to-one matching problems.

rium the desired matchings are obtained in spite of the fact that agents may behave strategically? The equilibrium notion we consider is the Nash equilibrium. Using the language of mechanism design, we are searching for (Nash) implementable solutions. [4]

Motivated by the negative results of Roth (1982b) and Alcalde and Barberà (1994), Kara and Sönmez (1996) search for implementable solutions for marriage problems. They show that the core correspondence is implementable. Furthermore, they show that any solution that is Pareto efficient, individually rational, and implementable is a supersolution of the core correspondence. That is, the core correspondence is the minimal implementable solution that is Pareto efficient and individually rational. [5] In this paper we generalize the results of Kara and Sönmez (1996) to the class of generalized matching problems. A corollary of these general results in the context of the housing market is that the core correspondence is the only single-valued solution that is Pareto efficient, individually rational, and implementable.

In this paper we show that we need to consider the core correspondence as a whole as long as we are is interested in implementation of Pareto-efficient and individually rational solutions to generalized matching problems. As a consequence, we also identify the loss entailed in obtaining implementability as well as Pareto efficiency and individual rationality: single-valuedness. We believe these results provide further non-cooperative support to the core correspondence, which is a cooperative solution.

2. Preliminaries

We divide this section into two subsections. Subsection 2.1 deals with implementation and related concepts in general mechanism design framework. Subsection 2.2 deals with generalized matching problems.

2.1. Implementation

The set of alternatives is A. The set of agents is $N=\{1, 2, \ldots, n\}$. For each agent $i \in N$, $\mathscr{R}_i$ is the set of possible preference relations. Here each $R_i \in \mathscr{R}_i$ is a *complete* (for all $a, b \in A$ we have aR_ib or bR_ia) and *transitive* (for all $a, b, c \in A$ we have aR_ib and bR_ic implies aR_ic) binary relation on A. Let $\mathscr{R} = \prod_{i \in N} \mathscr{R}_i$. The *lower contour set* of R_i at $a \in A$ is $L(a, R_i) = \{b \in A \mid aR_ib\}$.

[4] See Maskin (1985), Moore (1992), and Thomson (1993) for expositions of implementation theory.

[5] See also Alcalde (1996) and Mı (1994) for implementation results in marriage problems via refinements of the Nash equilibrium.

A *solution* is a correspondence $\varphi : \mathscr{R} \to A$. Here each alternative $a \in \varphi(R)$ is interpreted as a desirable allocation when the preference profile is R. A preference profile $\tilde{R}$ is obtained by a *monotonic transformation* of R at $a \in A$, if $L(a, R_i) \subseteq L(a, \tilde{R}_i)$ for all $i \in N$. Let $\mathrm{MT}(R, a)$ denote the set of preference profiles which are obtained by a monotonic transformation of R at a. A solution φ is *monotonic* if for all $R, \tilde{R} \in \mathscr{R}$, and for all $a \in \varphi(R)$, if $\tilde{R} \in \mathrm{MT}(R, a)$, then $a \in \varphi(\tilde{R})$. That is, a solution is monotonic if whenever an alternative a is selected for a preference profile R and the ranking of a improves for all agents under another preference profile $\tilde{R}$ (in the sense that no alternative that is weakly worse under R is strictly better under $\tilde{R}$) a is also selected under $\tilde{R}$. A solution φ satisfies *no veto power* if, for all $i \in N$, and for all $R \in \mathscr{R}$, if $A = L(a, R_j)$ for all $j \in N \setminus \{i\}$, then $a \in \varphi(R)$.

A *game form* is a pair $\Gamma = (X, h) = (\prod_{i \in N} X_i, h)$, where X_i is agent i's *strategy space*, and $h : X \to A$ is an *outcome function*. The pair (Γ, R) defines a *game*. Let $\mathrm{NE}(\Gamma, R)$ denote the set of pure strategy Nash equilibria for the game (Γ, R). The game form Γ *implements* the solution φ (in Nash equilibria), if $h(\mathrm{NE}(\Gamma, R)) = \varphi(R)$ for all $R \in \mathscr{R}$.

Maskin (1977) shows that monotonicity is a necessary condition for implementability. He further shows that monotonicity and no veto power together are sufficient for implementability. (See also Williams, 1986, and Saijo, 1988.) Recently there has been a number of studies identifying the necessary and sufficient conditions for implementability. Some of these studies are Moore and Repullo (1990), Dutta and Sen (1991), Sjöström (1991), Danilov (1992), and Yamato (1992). Here we present the results due to Danilov (1992) and Yamato (1992).

Let $\varphi : \mathscr{R} \to A$ and $B \subseteq A$. An alternative $b \in L(a, R_i)$ is *essential* for agent $i \in N$ for φ if

$$\exists R_i^* \in \mathscr{R}_i,\ L(b, R_i^*) \subseteq L(a, R_i) \text{ and } b \in \varphi(R^*).$$

That is, an alternative b in the lower contour set of R_i at a is essential for agent i for φ if we can find a preference profile R^* where any alternative that is strictly better than a under R_i is also better than b under R_i^* and b is selected for the preference profile R^*. We denote the set of essential alternatives for agent $i \in N$ in $L(a, R_i)$ for φ by $\mathrm{E}(\varphi, i, L(a, R_i))$. A rule φ is *essentially monotonic* if for all $R, \tilde{R} \in \mathscr{R}$ and for all $a \in \varphi(R)$, if $\mathrm{E}(\varphi, i, L(a, R_i)) \subseteq L(a, \tilde{R}_i)$ for all $i \in N$, then $a \in \varphi(\tilde{R})$. Thus a solution φ is essentially monotonic if whenever an alternative a is selected for a preference profile R and it is weakly preferred to all essential elements in $L(a, R_i)$ under $\tilde{R}$, it is selected for the preference profile $\tilde{R}$.

Danilov (1992) introduces the concept of essential monotonicity and shows that if $|N| \geq 3$, then a solution φ is implementable if and only if it is essentially monotonic. Danilov proves this result on a domain where preferences are linear orders on A. Yamato (1992) generalizes this result as follows. Let $\mathscr{R}$ be such that, for all $a \in A$, $R \in \mathscr{R}$, $i \in N$, and $b \in L(a, R_i)$, there exists $R' \in \mathscr{R}$ such that

$L(b, R'_i) = L(a, R_i)$ and for all $j \neq i$, $L(b, R'_j) = A$. Then, if $|N| \geq 3$, a solution φ is implementable if and only if it is essentially monotonic.

2.2. Generalized matching problems

A *(generalized) matching problem* is a triple $G = (N, S, R)$. The first component, N, is a finite set of agents. The second component, $S = (S_i)_{i \in N}$, is a list of subsets of N with $i \in S_i$ for all $i \in N$. Here S_i represents the set of possible assignments for agent i. The last component, $R = (R_i)_{i \in N}$, is a list of preference relations. Let P_i denote the strict relation associated with the preference relation R_i for all $i \in N$. The preference relation R_i of each agent $i \in N$ is *reflexive* (for all $j \in S_i$ we have jR_ij), *transitive*, and *total* (for all j, $k \in S_i$ with $j \neq k$ we have jR_ik or kR_ij, but not both). Such preference relations are referred to as *linear orders*. Let $\mathscr{R}_i$ be the class of linear orders on S_i and $\mathscr{R} = \Pi_{i \in N}\mathscr{R}_i$. We consider the case where N and S are fixed, and hence to define a matching problem it suffices to specify a preference profile.

A *(generalized) matching* μ is a function from the set N into itself such that

(1) $\forall i \in N$, $\mu(i) \in S_i$,

(2) $\forall i \in N$, $|\mu^{-1}(i)| = 1$.

Note that μ is a bijection on N. For all $i \in N$, we refer to $\mu(i)$ as the *assignment* of i at μ. We denote the set of all matchings by $\mathscr{M}$. Let $\mu_I \in \mathscr{M}$ be defined by $\mu_I(i) = i$ for all $i \in N$. We exogenously specify a subset $\mathscr{M}^{\mathrm{f}}$ of the set of matchings $\mathscr{M}$ as the set of *feasible matchings*. We always require that $\mu_I \in \mathscr{M}^{\mathrm{f}}$. In the context of matching problems the set of allocations A is the set of feasible matchings $\mathscr{M}^{\mathrm{f}}$. Given a preference relation R_i of an agent $i \in N$, initially defined over S_i, we extend it to the set of feasible matchings $\mathscr{M}^{\mathrm{f}}$ in the following natural way: agent i prefers the matching μ to the matching μ' if and only if he prefers his assignment under μ to his assignment under μ'. We slightly abuse the notation and also use R_i to denote this extension.

Two extensively studied subclasses of generalized matching problems are the *housing market* (Shapley and Scarf, 1974) and the *marriage problems* (Gale and Shapley, 1962). In the housing market, each agent owns one house and has preferences over the houses held by all agents. An allocation is a permutation of the houses among the agents. In the marriage problems, there are two sets of agents: the set of men M and the set of women W. Each man has preferences over the set of women and staying single. Similarly, each woman has preferences over the set of men and staying single. An allocation here is a matching of men and women (where agents may end up being single). If we specify $S_i = N$ for all $i \in N$ and $\mathscr{M}^{\mathrm{f}} = \mathscr{M}$ we obtain the housing market as a subclass of generalized matching problems. If we specify $N = M \cup W$, where M and W are two finite, non-empty, disjoint sets, $S_m = W \cup \{m\}$ for all $m \in M$, $S_w = M \cup \{w\}$ for all $w \in W$, and

$$\mathscr{M}^{\mathrm{f}} = \{\mu \in \mathscr{M} \mid \mu(\mu(i)) = i, \text{ for all } i \in N\},$$

we obtain the marriage problems as a subclass of generalized problems.

A matching $\mu \in \mathscr{M}^{\mathrm{f}}$ is *individually rational* under R if $\mu(i) R_i i$ for all $i \in N$. We denote the set of all individually rational matchings under R by $\mathscr{I}(R)$.

A matching $\mu \in \mathscr{M}^{\mathrm{f}}$ is *Pareto efficient* under R if there is no other matching $\mu' \in \mathscr{M}^{\mathrm{f}}$ such that $\mu'(i) R_i \mu(i)$ for all $i \in N$ and $\mu'(j) P_j \mu(j)$ for some $j \in N$. We denote the set of all Pareto-efficient matchings under R by $\mathscr{P}(R)$.

A matching $\mu' \in \mathscr{M}^{\mathrm{f}}$ *dominates* the matching $\mu \in \mathscr{M}^{\mathrm{f}}$ via a coalition $T \subseteq N$ under R if

(1) $\forall i \in T,\ \mu'(i) \in T$;

(2) $\forall i \in T,\ \mu'(i) R_i \mu(i)$,

(3) $\exists j \in T,\ \mu'(j) P_j \mu(j)$.

In that case we say the coalition T *blocks* μ under R. A matching $\mu \in \mathscr{M}^{\mathrm{f}}$ is in the *core* of the matching problem $R \in \mathscr{R}$ if it is not dominated by any matching. We denote the core of R by $\mathscr{C}(R)$. In the context of matching problems we refer to solutions as matching rules. A matching rule φ is *Pareto efficient* if $\varphi(R) \subseteq \mathscr{P}(R)$ for all $R \in \mathscr{R}$, and *individually rational* if $\varphi(R) \subseteq \mathscr{I}(R)$ for all $R \in \mathscr{R}$.

3. Results

Throughout this paper we assume that N, S, and $\mathscr{M}^{\mathrm{f}}$ are such that the core is non-empty for all preference profiles. Let $\mathscr{C}$ be the matching rule which selects the set of matchings in the core for each preference profile. We will refer to the matching rule $\mathscr{C}$ as the core correspondence. The first proposition concerns the monotonicity of the core correspondence.

Proposition 1. The core correspondence is monotonic.

Proof. Suppose $\mathscr{C}$ is not monotonic. Then there exists R, $\tilde{R} \in \mathscr{R}$ and $\mu \in \mathscr{C}(R)$ with $L(\mu, R_i) \subseteq L(\mu, \tilde{R}_i)$ for all $i \in N$ but $\mu \notin \mathscr{C}(\tilde{R})$. Hence there exists $T \subseteq N$ and $\mu' \in \mathscr{M}^{\mathrm{f}}$ such that

(1) $\forall i \in T,\ \mu'(i) \in T$,

(2) $\forall i \in T,\ \mu'(i) \tilde{R}_i \mu(i)$,

(3) $\exists j \in T,\ \mu'(j) \tilde{P}_j \mu(j)$.

This implies $\mu'(i) R_i \mu(i)$ for all $i \in T$ as $L(\mu, R_i) \subseteq L(\mu, \tilde{R}_i)$ for all $i \in N$. We also have $\mu'(j) \neq \mu(j)$ and the preferences are strict. Therefore $\mu'(j) R_j \mu(j)$ implies $\mu'(j) P_j \mu(j)$ and therefore

(1) $\forall i \in T,\ \mu'(i) \in T$,

(2) $\forall i \in T,\ \mu'(i) R_i \mu(i)$,

(3) $\exists j \in T,\ \mu'(j) P_j \mu(j)$,

which contradicts $\mu \in \mathscr{C}(R)$. □Q.E.D.

Our first theorem concerns monotonic rules that are Pareto efficient and individually rational.

Theorem 1. Let φ be a Pareto-efficient, individually rational, and monotonic matching rule. Then $\varphi \supseteq \mathscr{C}$.

Proof. Let $R \in \mathscr{R}$ and $\mu \in \mathscr{C}(R)$. We need to show that $\mu \in \varphi(R)$. Let $R' \in \mathscr{R}$ be such that for all $i \in N$,

(1) $jP'_i k \Leftrightarrow jP_i k$, for all $j, k \in S_i \setminus \{i\}$,

(2) $\mu(i) R'_i i$ and $\nexists j \in S_i \setminus \{i, \mu(i)\}$ with $\mu(i) R'_i j R'_i i$.

Note that $R' \in \mathrm{MT}(R, \mu)$ and $\mu \in \mathscr{C}(R)$. Therefore $\mu \in \mathscr{C}(R')$ by Proposition 1. We also have $R \in \mathrm{MT}(R', \mu)$.

Let $\nu \in \mathscr{M}^{\mathrm{f}}$ be such that $\nu \in \mathscr{I}(R')$. Let $i_1 \in N$. Let $|N| = n$. Let $i_{k+1} = \nu(i_k)$ for all $k \in \{1, 2, \ldots, n\}$.

Let us suppose

$$i_2 P'_{i_1} \mu(i_1). \tag{1}$$

We show that

$$\nu(i_k) \notin \{i_1, i_2, \ldots, i_k\}, \text{ for all } k \in \{2, 3, \ldots, n\}$$

by induction on k. Let us first show that $\nu(i_2) \notin \{i_1, i_2\}$. We have $i_2 \notin \{i_1, \mu(i_1)\}$ by relation (1) and the construction of R'_{i_1}. Therefore, $\nu(i_2) \neq i_2$ since ($i_2 \neq i_1$, $\nu(i_1) = i_2$, and $|\nu^{-1}(i_2)| = 1$).

We either have $\nu(i_2) = i_1$ or $\nu(i_2) \neq i_1$. If the former holds, then $\nu \in \mathscr{I}(R')$ implies $i_1 P'_{i_2} i_2$ and hence

$$i_1 R'_{i_2} \mu(i_2) \tag{2}$$

by the construction of R'_{i_2}. But then the coalition $\{i_1, i_2\}$ blocks μ under R' by relations (1) and (2), which contrads $\mu \in \mathscr{C}(R')$. Therefore $\nu(i_2) \notin \{i_1, i_2\}$.

Next, let us suppose that $\nu(i_k) \notin \{i_1, i_2, \ldots, i_k\}$ for all $k \in \{2, 3, \ldots, l\}$ with $2 \leq l < n$. Then we have

$$\nu(i_k) = i_{k+1} \neq i_k, \quad \text{for all } k \in \{2, 3, \ldots, l\}.$$

Therefore

$$\nu(i_k) = i_{k+1} P'_{i_k} i_k, \quad \text{for all } k \in \{2, 3, \ldots, l\}$$

as $\nu(R') \in \mathscr{I}(R')$, and hence

$$\nu(i_k) = i_{k+1} R'_{i_k} \mu(i_k), \quad \text{for all } k \in \{2, 3, \ldots, l\} \tag{3}$$

by construction. We have $i_{l+1} = \nu(i_l) \notin \{i_1, i_2, \ldots, i_l\}$. But $\nu(i_k) = i_{k+1}$ for all $k \in \{1, 2, \ldots, l\}$ and ν is a bijection, therefore $\nu(i_{l+1}) \notin \{i_2, \ldots, i_{l+1}\}$. We either have $\nu(i_{l+1}) = i_1$ or $\nu(i_{l+1}) \neq i_1$. If the former holds, then $\nu \in \mathscr{I}(R')$ implies $i_1 P'_{i_{l+1}} i_{l+1}$ and hence

$$i_1 R'_{i_{l+1}} \mu(i_{l+1}) \tag{4}$$

by the construction of $R'_{i_{l+1}}$. But then the coalition $\{i_1, i_2, \ldots, i_{l+1}\}$ blocks μ under R' by relations (1), (3), and (4), which contradicts $\mu \in \mathscr{C}(R')$. Therefore $\nu(i_{l+1}) \notin \{i_1, i_2, \ldots, i_{l+1}\}$. Hence $\nu(i_n) \notin \{i_1, i_2, \ldots, i_n\}$ by induction.

We have

$$i_2 = \nu(i_1) \notin \{i_1\},$$
$$i_3 = \nu(i_2) \notin \{i_1, i_2\},$$
$$\vdots$$
$$i_n = \nu(i_{n-1}) \notin \{i_1, i_2, \ldots, i_{n-1}\}.$$

Therefore $i_j \neq i_k$ for all $j, k \in \{1, 2, \ldots, n\}$ with $j \neq k$, which implies $\{i_1, i_2, \ldots, i_n\} = N$. Thus, $\nu(i_n) \notin N$, which contradicts $\nu \in \mathscr{M}^{\mathrm{f}}$. Hence $\mu(i_1) R'_{i_1} \nu(i_1) = i_2$. That is,

$$\forall i \in N,\ \nu \in \mathscr{I}(R'),\ \mu(i) R'_i \nu(i),$$

and therefore

$$\mathscr{P}(R') \cap \mathscr{I}(R') = \{\mu\},$$

which implies

$$\varphi(R') = \{\mu\}.$$

We also have $R \in \mathrm{MT}(R', \mu)$ and φ is monotonic, therefore $\mu \in \varphi(R)$. □Q.E.D.

Remark 1. Theorem 1 also hold for cases where the core correspondence is not well-defined in the sense that any rule that is Pareto efficient, individually rational, and implementable should select all the matchings in the core whenever it is non-empty. One such class of problems is the *roommate problems* (Gale and Shapley, 1962): there is a group of agents each of whom has strict preferences over all agents. An allocation is a partition of the set of agents into groups of size one and two. Here we assign either one or two persons to a room. We obtain roommate problems as generalized matching problems as follows. Let $S_i = N$ for all $i \in N$ and

$$\mathscr{M}^{\mathrm{f}} = \{\mu \in \mathscr{M} \mid \mu(\mu(i)) = i, \text{ for all } i \in N\}.$$

Let us consider the following example: $N = \{i, j, k\}$, $jP_i kP_i i$, $kP_j iP_j j$, $iP_k jP_k k$. Note that in this problem staying single is each agent's last choice and each agent is someone else's first choice. Therefore whoever stays single in a matching will form a coalition to block this matching. Hence $\mathscr{C}(R) = \emptyset$. It is straightforward to construct roomate problems with a non-empty core.

Theorem 1 shows that if we have any hope of implementing a Pareto-efficient and individually rational matching rule, it is the core and its supersolutions. The next natural question is: Is the core correspondence implementable? The core correspondence is monotonic by Proposition 1, yet it does not satisfy no veto power. Hence we need to refer to Danilov (1992) and Yamato (1992) to answer this question. Using the tools developed in these papers we can show that the core correspondence is implementable. Before stating and proving the theorem, we have the following lemma.

Lemma 1. For all $R \in \mathscr{R}$, $\mu \in \mathscr{C}(R)$, *and* $i \in N$ *we have* $\mathrm{E}(\mathscr{C}, i, L(\mu, R_i)) = L(\mu, R_i)$.

Proof. Let $R \in \mathscr{R}$, $\mu \in \mathscr{C}(R)$, $i \in N$. Let $\mu' \in \mathrm{E}(\mathscr{C}, i, L(\mu, R_i))$. Then there exists a preference profile $R' \in \mathscr{R}$ such that $L(\mu', R'_i) \subseteq L(\mu, R_i)$ and therefore $\mu' \in L(\mu, R_i)$. Hence

$$\mathrm{E}(\mathscr{C}, i, L(\mu, R_i)) \subseteq L(\mu, R_i). \tag{5}$$

Next, let $\mu' \in L(\mu, R_i)$. Let $R' \in \mathscr{R}$ be such that

(1) (a) $\mu'(i) R'_i i$,
   (b) $\forall j \in S_i \setminus \{i\}$, $j R'_i \mu'(i)$.
(2) $\forall k \in N \setminus i$
   (a) $\mu'(k) R'_k k$,
   (b) $\forall l \in S_k \setminus \{\mu'(k)\}$, $k R'_k l$.

We have $\mu' \in \mathscr{C}(R')$ and for all $\tilde{\mu} \in L(\mu', R'_i)$, $\tilde{\mu}(i) \in \{\mu'(i), i\}$. Therefore $\mu(i) R_i \tilde{\mu}(i)$ or, equivalently, $\tilde{\mu} \in L(\mu, R_i)$ and hence $L(\mu', R'_i) \subseteq L(\mu, R_i)$. This, together with $\mu \in \mathscr{C}(R')$, implies that $\mu' \in \mathrm{E}(\mathscr{C}, i, L(i, R_i))$. Therefore

$$L(\mu, R_i) \subseteq \mathrm{E}(\mathscr{C}, i, L(\mu, R_i)). \tag{6}$$

Relations (5) and (6) imply the desired result. □Q.E.D.

Theorem 2. Let $|N| \geq 3$. *Then the core correspondence is implementable.*

Proof. Lemma 1 with monotonicity of the core correspondence (Proposition 1) implies that the core is essentially monotonic. Therefore the core correspondence is implementable, by Yamato (1992). □Q.E.D.

Remark 2. Kara and Sönmez (1996) show that the core correspondence is not implementable on the class of marriage problems whenever $|N| = 2$. As negative results are stronger in smaller classes, this result extends to generalized matching problems.

These results have an interesting implication for the housing market.

Corollary 1. Consider the housing market. The core correspondence is the only single-valued matching rule that is Pareto efficient, individually rational, and implementable.

Proof. Roth and Postlewaite (1977) show that the core correspondence is single-valued in the context of the housing market. This, together with Theorem 1 and Theorem 2, implies the desired result. □ Q.E.D.

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