EC 308 Question Set # 1

1. Consider the following game: There are 2 steps. In Step 1 Player 1 chooses between throwing 1 unit of his own payoff (strategy T) or not (strategy N). Observing his action in Step 2 they simultaneously play the following game:

		Player 2	
		L	R
Player 1	U	4,2	1,1
	D	1,1	2,4

   Represent this scenario as an extensive form game. Do not forget to adjust Player 1's payoff in case he decides to throw 1 unit of his payoff. Do not solve the game.

2. Player 1 and Player 2 play the following two step game: In the first step they play the matching pennies game where they simultaneously choose between H and T each. If there is a match Player 1 is the winner and they play the following simultaneous game in the second step:

		Player 2	
		L	R
Player 1	U	5,1	0,0
	D	0,0	3,1

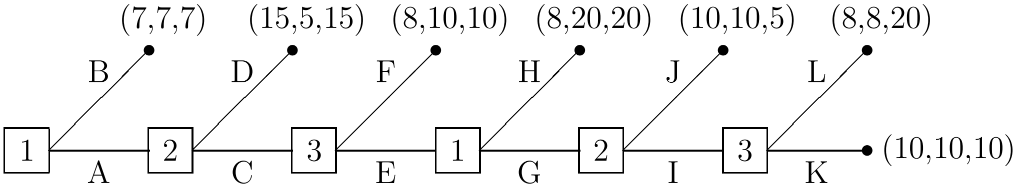
   If there is no match Player 2 is the winner and they play the following simultaneous game in the second step:

		Player 2	
		L	R
Player 1	U	1,5	0,0
	D	0,0	1,3

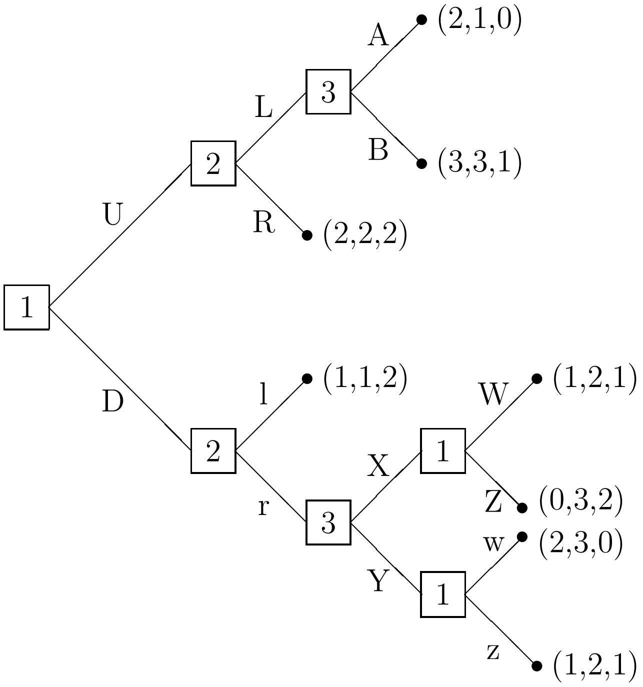
   Represent this scenario as an extensive form game. Do not solve the game. (Hint: First remember how to represent simultaneous games as extensive form games with the help of information sets.)

3. Solve the following games with backwards induction. Give the equilibrium strategies as well as equilibrium payoffs.

   (a)



(b)



4. Find the dominant and dominated strategies in the following games (in case players have such strategies). Is there any dominant strategy equilibrium in any of these games?

(a)

Player 2

		L	R
Player 1	U	5,5	3,6
	D	6,3	4,4

(b)

P3

A

P2

		L	R
	U	5,5,1	1,4,0
P1	M	6,4,5	2,5,3
	D	3,1,2	3,3,1

B

P2

	L	R
U	4,8,0	2,1,0
M	5,4,3	3,3,3
D	3,5,1	5,3,0

(c)

		Player 2				
		V	W	X	Y	Z
	A	2,5	4,4	5,5	0,4	1,5
	B	5,3	3,3	4,3	2,2	1,0
Player 1	C	6,0	3,0	1,1	1,1	0,1
	D	6,2	3,0	4,2	3,2	1,2
	E	1,1	5,1	4,2	1,0	0,2

5. Solve the following games with iterated elimination of dominated strategies. Indicate the order of elimination and at each step explain why these strategies are eliminated (i.e. indicate which strategy dominates dominate them).

(a)

		Player 2	
		L	R
Player 1	U	5,3	0,4
	D	0,3	2,3

(b)

		P2	
		L	R
	U	4,1	6,2
P1	M	6,1	5,2
	D	5,5	5,1

(c)

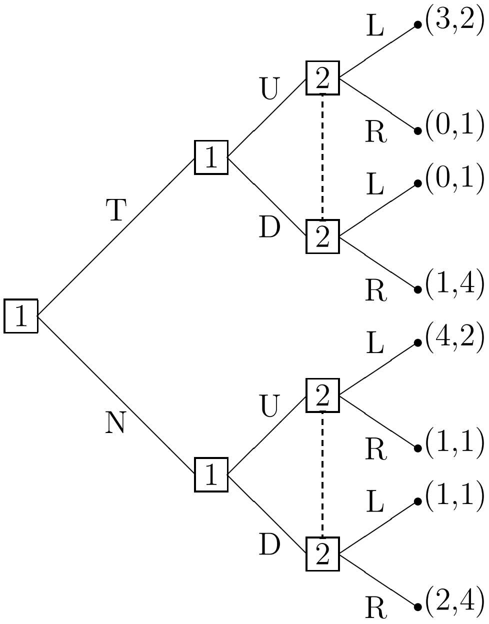
		Player 2				
		V	W	X	Y	Z
	A	3,3	1,4	5,4	1,2	2,5
	B	4,1	1,1	3,5	1,4	1,2
Player 1	C	0,5	2,4	3,0	1,7	1,1
	D	5,0	1,2	4,2	1,4	5,1
	E	1,1	2,3	3,2	2,1	2,3

Question Set # 1 - Answer Key

1. *Consider the following game: There are 2 steps. In Step 1 Player 1 chooses between throwing 1 unit of his own payoff (strategy T) or not (strategy N). Observing his action in Step 2 they simultaneously play the following game:*

		Player 2	
		L	*R*
Player 1	*U*	*4,2*	*1,1*
	D	*1,1*	*2,4*

   Represent this scenario as an extensive form game. Do not forget to adjust Player 1's payoff in case he decides to throw 1 unit of his payoff. Do not solve the game.



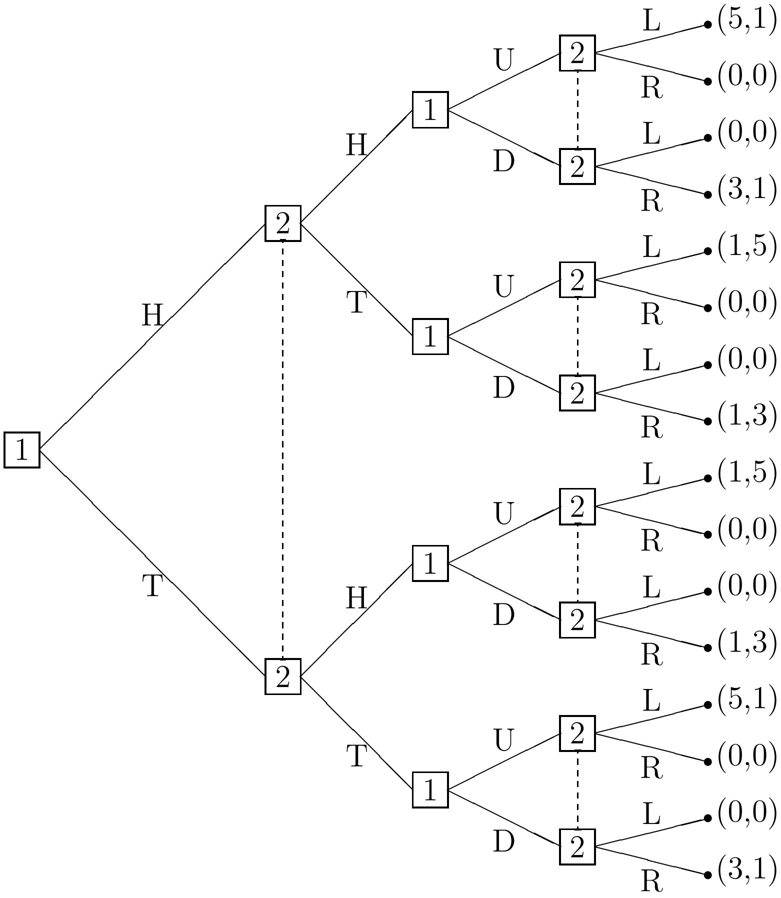
2. *Player 1 and Player 2 play the following two step game: In the first step they play the matching pennies game where they simultaneously choose between H and T each. If there is a match Player 1 is the winner and they play the following simultaneous game in the second step:*

		Player 2	
		L	*R*
Player 1	*U*	*5,1*	*0,0*
	D	*0,0*	*3,1*

   If there is no match Player 2 is the winner and they play the following simultaneous game in the second step:

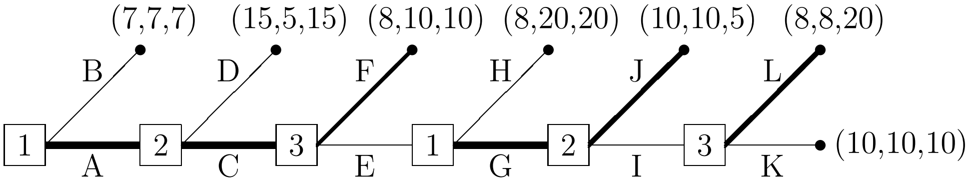
		Player 2	
		L	*R*
Player 1	*U*	*1,5*	*0,0*
	D	*0,0*	*1,3*

Represent this scenario as an extensive form game. Do not solve the game. (Hint: First remember how to represent simultaneous games as extensive form games with the help of information sets.)



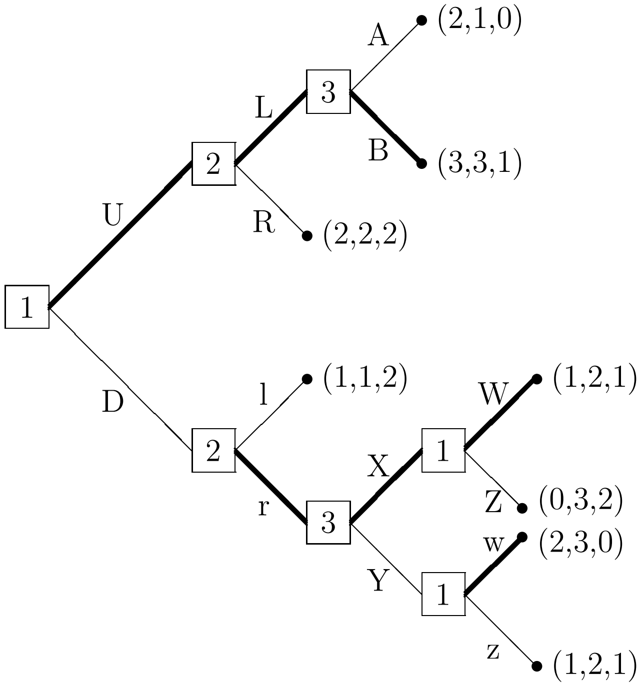
3. *Solve the following games with backwards induction. Give the equilibrium strategies as well as equilibrium payoffs.*

   (a)



The thick lines on the picture above represent the solution of the game following from the backward induction. The equilibrium strategies (we need to specify what each player would do in any situation) are $((A,G),(C,J),(F,L))$ and the equilibrium payoff is (8,10,10).

(b)



The equilibrium strategies are: $((U,W,w),(L,r),(B,X))$. The equilibrium payoff is (3,3,1).

4. *Find the dominant and dominated strategies in the following games (in case players have such strategies). Is there any dominant strategy equilibrium in any of these games?*

   (a)

		Player 2	
		L	R
Player 1	U	5,5	3,6
	D	6,3	4,4

   There are 2 dominated strategies: L is dominated by R (R always yields better payoff for player 2 than L) and U is dominated by D (D always yields higher payoff for player 1 than U). By the same token, D and R are dominant strategies - D is always the best choice for player 1, R is always the best choice for player 2. Since both players have dominant strategies, there is a dominant strategy equilibrium — (D, R). The equilibrium payoffs are $(4,4)$.

(b)

P3

A

P2

		L	R
	U	5,5,1	1,4,0
P1	M	6,4,5	2,5,3
	D	3,1,2	3,3,1

B

P2

	L	R
U	4,8,0	2,1,0
M	5,4,3	3,3,3
D	3,5,1	5,3,0

U is dominated by M (M yields a higher payoff for player 1 no matter what two other players do). Strategy B is dominated by A — player 3 always prefers payoff in the 'left' table (following playing A) than in the corresponding cell in the 'right' table (following move B). Since player 3 has only two strategies, A is also a dominant strategy. Player 2 doesn't have any dominant or dominated strategies — e.g. when other players play (U,B) playing L is a better choice than playing R, when other players choose (M,A), R is a better choice. So none of the strategies dominates the other.

Players 1 and 2 don't have dominant strategies so there is no dominant strategy equilibrium.

(c)

Player 2

		V	W	X	Y	Z
	A	2,5	4,4	5,5	0,4	1,5
	B	5,3	3,3	4,3	2,2	1,0
Player 1	C	6,0	3,0	1,1	1,1	0,1
	D	6,2	3,0	4,2	3,2	1,2
	E	1,1	5,1	4,2	1,0	0,2

Player 1's dominated strategies: B (dominated by D) and C (dominated by D). Player 1 has no dominant strategies.

Player 2's dominated strategies: V, W, Y, Z (all dominated by X). X is a dominant strategy.

There is no dominant strategy equilibrium because player 1 doesn't have a dominant strategy.

5. *Solve the following games with iterated elimination of dominated strategies. Indicate the order of elimination and at each step explain why these strategies are eliminated (i.e. indicate which strategy dominates dominate them).*

(a)

		Player 2	
		L	R
Player 1	U	5,3	0,4
	D	0,3	2,3

Strategy L is dominated by R (so we can eliminate L) and after removing strategy L, U is dominated by D. So we have a single remaining solution (D,R) and a corresponding payoff (2,3).

(b)

		P2	
		L	R
	U	4,1	6,2
P1	M	6,1	5,2
	D	5,5	5,1

Strategy D is dominated by strategy M. After eliminating D, strategy L is dominated by R (playing L, player 2 gets always payoff equal to 1, playing R he gets payoff 2). When we eliminate L, strategy M will be dominated by U. At this point the only remaining choice is (U,R) and equilibrium payoff is (6,2).

(c)

		Player 2				
		V	W	X	Y	Z
	A	3,3	1,4	5,4	1,2	2,5
	B	4,1	1,1	3,5	1,4	1,2
Player 1	C	0,5	2,4	3,0	1,7	1,1
	D	5,0	1,2	4,2	1,4	5,1
	E	1,1	2,3	3,2	2,1	2,3

We can notice that strategy C is dominated by strategy E (player 1 always gets at least as much playing E as playing C). So we can eliminate strategy E. Then V is dominated by any of the strategies W, X, Z. The game that remains after eliminating V is:

	W	X	Y	Z
A	1,4	5,4	1,2	2,5
B	1,1	3,5	1,4	1,2
D	1,2	4,2	1,4	5,1
E	2,3	3,2	2,1	2,3

One can see that in this game strategy B is dominated by strategy A and, after eliminating B, X is dominated by W.

	W	Y	Z
A	1,4	1,2	2,5
D	1,2	1,4	5,1
E	2,3	2,1	2,3

The table above represents the game after eliminating strategies B and X. We can see now that A is dominated by E (so that we can eliminate A) and after doing so Z is dominated by W. What remains is:

	W	Y
D	1,2	1,4
E	2,3	2,1

Now E dominates D and W dominates Y (after eliminating D) so we are left with a solution (E,W) and equilibrium payoff (2,3).

EC 308 Question Set # 2

1. Find all the Nash equilibria of the following games. (Please indicate both equilibrium strategies and payoffs.)

   (a)

		Player 2	
		X	Y
	A	2,3	0,4
Player 1	B	2,2	1,1
	C	1,1	1,1
	D	0,1	0,3

   (b)

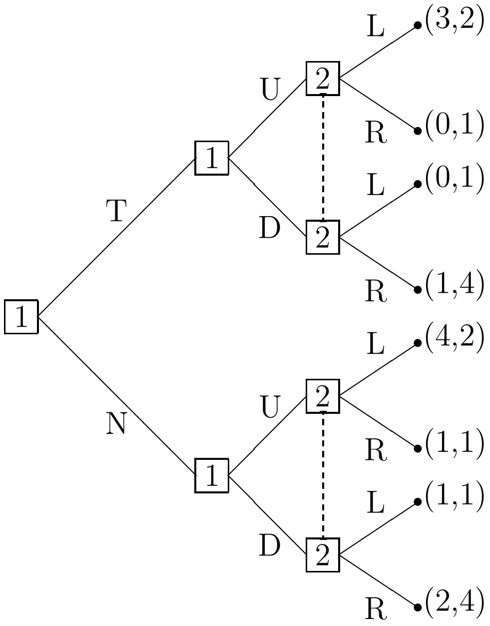
P3

		A			B	
		P2			P2	
		L	R		L	R
	U	5,5,0	1,4,0	U	4,8,0	2,1,0
P1	M	6,4,5	2,5,3	M	4,1,3	3,3,3
	D	3,1,2	3,3,1	D	3,5,1	0,3,0

   (c)

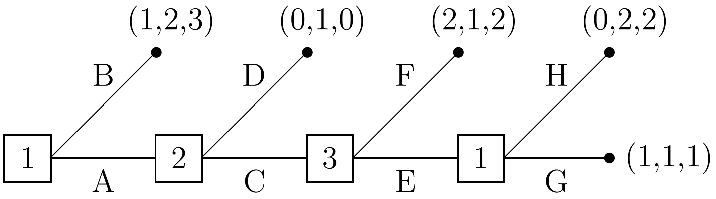
		Player 2				
		V	W	X	Y	Z
	A	2,5	4,4	5,5	0,4	1,5
	B	5,3	3,3	4,3	2,2	1,0
Player 1	C	6,0	3,0	1,1	1,1	0,1
	D	6,2	3,0	4,2	3,2	1,2
	E	1,1	5,1	4,2	1,0	0,2

2. Recall that we obtained the following extensive form game for the "throwing the payoff" scenario in question set 1:



   (a) Find the equivalent strategic game of this extensive form game.
   (b) Iteratively eliminate the dominated strategies in the strategic game you obtained in part a. (Here at each step eliminate as many strategies as you can.) Indicate the order of the elimination. Eventually two strategy-profiles (i.e. two cells in the strategic game) will survive. Which ones? What are their payoffs?

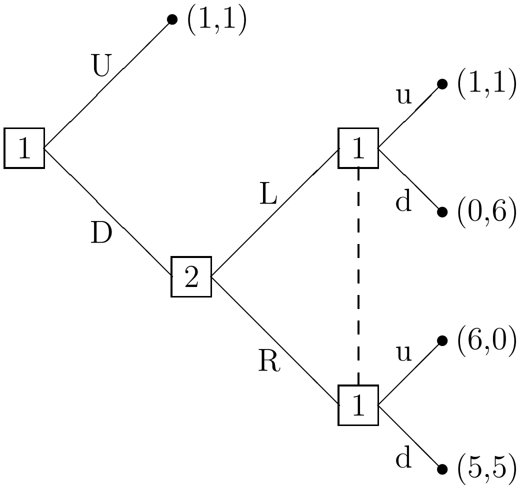
3. Consider the following extensive form game:



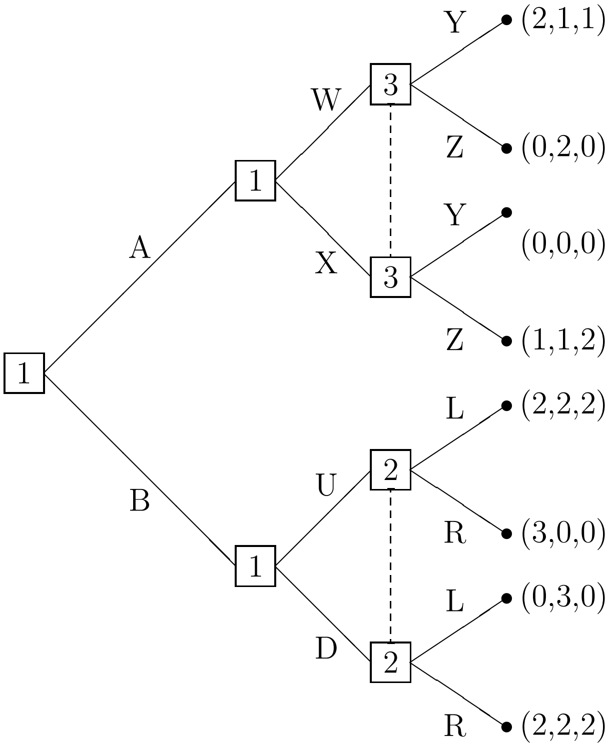
   (a) Find its equivalent strategic form game.
   (b) Find all its Nash equilibria using the equivalent strategic form game. (Please give equilibrium strategies as well as payoffs.)
   (c) Find all its subgame perfect Nash equilibria. (As always, provide strategies as well as payoffs.) Are there any Nash equilibria that are not subgame perfect? If so, which ones?

4. Find all the subgame perfect Nash equilibria of the following games. Please give equilibrium strategies as well as payoffs.

(a)


1
U
(1,1)
D
2
L
R
1
1
u
(1,1)
d
(0,6)
u
(6,0)
d
(5,5)

(b)


1
A
B
1
W
X
3
3
Y
(2,1,1)
Z
(0,2,0)
Y
(0,0,0)
Z
(1,1,2)
1
U
D
2
2
L
(2,2,2)
R
(3,0,0)
L
(0,3,0)
R
(2,2,2)

Question Set # 2 - Answer Key

1. *Find all the Nash equilibria of the following games. (Please indicate both equilibrium strategies and payoffs.)*

   (a)

		Player 2	
		X	Y
	A	2,3	0,4
Player 1	B	2,2	1,1
	C	1,1	1,1
	D	0,1	0,3

   There are two Nash equilibria: (B,X) with the payoffs of (2,2) and (C,Y) with the payoffs (1,1).

   (b)

P3

A

		P2	
		L	R
	U	5,5,0	1,4,0
P1	M	6,4,5	2,5,3
	D	3,1,2	3,3,1

B

	P2	
	L	R
U	4,8,0	2,1,0
M	4,1,3	3,3,3
D	3,5,1	0,3,0

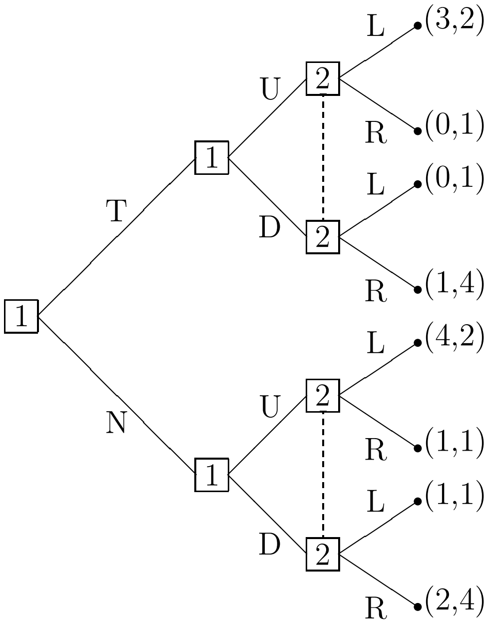
   There are three Nash equilibria in this game: (U,L,B) with payoffs (4,8,0), (M,R,B) with payoffs (3,3,3) and (D,R,A) with payoffs (3,3,1).

   (c)

		Player 2				
		V	W	X	Y	Z
	A	2,5	4,4	5,5	0,4	1,5
	B	5,3	3,3	4,3	2,2	1,0
Player 1	C	6,0	3,0	1,1	1,1	0,1
	D	6,2	3,0	4,2	3,2	1,2
	E	1,1	5,1	4,2	1,0	0,2

   There are 5 Nash equilibria: (A,X) - payoffs (5,5), (A,Z) - payoffs (1,5), (D,V) - payoffs (6,2), (D,Y) - payoffs (3,2) and (D,Z) - payoffs (1,2).

2. *Recall that we obtained the following extensive form game for the "throwing the payoff" scenario in question set 1:*



(a) *Find the equivalent strategic game of this extensive form game.*

The strategy of player 1 must specify what he is going to do at the initial node and each of the two subgames (so it consists of three moves). Strategy of player 2 specifies his moves in each of the subgames.

Player 2

		LL	L R	RL	RR
	TUU	3,2	3,2	0,1	0,1
	TUD	3,2	3,2	0,1	0,1
	TDU	0,1	0,1	1,4	1,4
Player 1	TDD	0,1	0,1	1,4	1,4
	NUU	4,2	1,1	4,2	1,1
	NUD	1,1	2,4	1,1	2,4
	NDU	4,2	1,1	4,2	1,1
	NDD	1,1	2,4	1,1	2,4

(b) *Iteratively eliminate the dominated strategies in the strategic game you obtained in part a. (Here at each step eliminate as many strategies as you can.) Indicate the order of the elimination. Eventually two strategy-profiles (i.e. two cells in the strategic game) will survive. Which ones? What are their payoffs?* First observe that strategies TDU and TDD are both dominated by NUU (and also by NDU) for Player 1, so one can eliminate them both:

	LL	LR	RL	RR
TUU	3,2	3,2	0,1	0,1
TUD	3,2	3,2	0,1	0,1
NUU	4,2	1,1	4,2	1,1
NUD	1,1	2,4	1,1	2,4
NDU	4,2	1,1	4,2	1,1
NDD	1,1	2,4	1,1	2,4

Now one can see that Player 2's strategy RL is dominated by LL and that strategy RR is dominated by LR:

	LL	LR
TUU	3,2	3,2
TUD	3,2	3,2
NUU	4,2	1,1
NUD	1,1	2,4
NDU	4,2	1,1
NDD	1,1	2,4

In the remaining game NUD and NDD are dominated by TUU (and TUD) for Player 1, and after eliminating them LL dominates LR for Player 2 so that we can also remove LR:

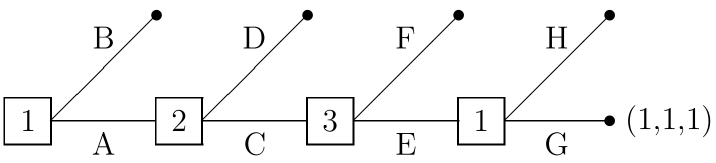
	LL
TUU	3,2
TUD	3,2
NUU	4,2
NDU	4,2

now Player 1 will never choose to play TUU or TUD because they are dominated by both NUU and NDU and we are left with:

	LL
NUU	4,2
NDU	4,2

so the surviving strategy profiles are (NUU, LL) and (NDU, LL) and they both offer payoffs of (4,2).

3. *Consider the following extensive form game:*



(a) *Find its equivalent strategic form game.*

		P3			
		E		F	
		P2		P2	
		C	D	C	D
	AG	1,1,1	0,1,0	2,1,2	0,1,0
P1	AH	0,2,2	0,1,0	2,1,2	0,1,0
	BG	1,2,3	1,2,3	1,2,3	1,2,3
	BH	1,2,3	1,2,3	1,2,3	1,2,3

(b) *Find all its Nash equilibria using the equivalent strategic form game. (Please give equilibrium strategies as well as payoffs.)*

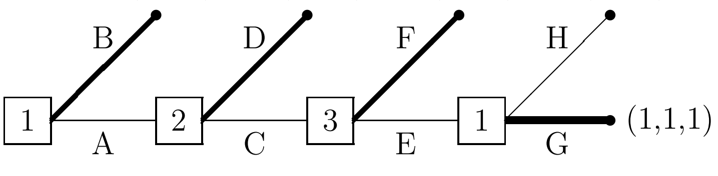
There are a number of Nash equilibria:

1. (AG,C,F) payoffs (2,1,2)
2. (AH,C,F) payoffs (2,1,2)
3. (BG,C,E) payoffs (1,2,3)
4. (BG,D,E) payoffs (1,2,3)
5. (BH,C,E) payoffs (1,2,3)
6. (BH,D,E) payoffs (1,2,3)
7. (BG,D,F) payoffs (1,2,3)
8. (BH,D,F) payoffs (1,2,3)

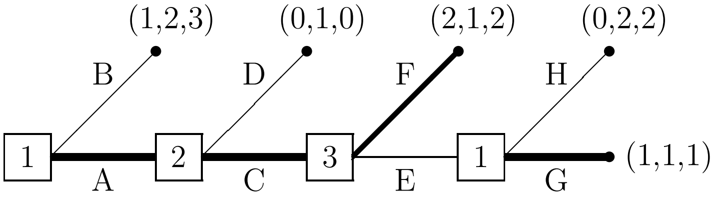
(c) *Find all its subgame perfect Nash equilibria. (As always, provide strategies as well as payoffs.) Are there any Nash equilibria that are not subgame perfect? If so, which ones?*

To find the subgame perfect equilibria we can just use the backward induction (because it's a sequential game). In the last stage, player 1 will always choose G (it gives a higher payoff than H) — it also means that Nash equilibria 2,5,6 and 8 are not subgame perfect. Given this information, player 3 prefers F to E (F offers payoff of 2, E pays 1) — thus also equilibria 3 and 4 are not subgame perfect. Now player 2 is indifferent between C and D — each of them pays 1. So we need to consider two cases:

Case 1 (Player 2 plays D): Player 1 chooses B in the first stage (the payoff is 1 instead of 0) - and this is one of the subgame perfect Nash equilibria: (BG, D, F) with payoffs (1,2,3).



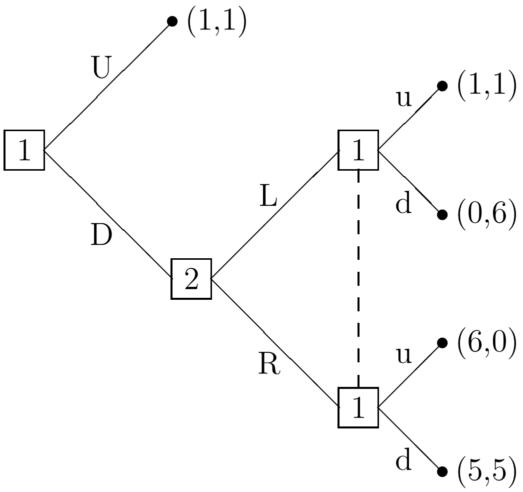
Case 2 (Player 2 plays C): Then Player 1 decides to play A — payoff is 2 (because game ends with move F), playing B instead would pay only 1. This is the second subgame perfect Nash equilibrium: (AG, C, F) with payoffs (2,1,2).



As we can see not all of the Nash equilibria are subgame perfect. Equilibria 2,3,4,5,6 and 8 (found in part b of the problem) are not equilibria in every subgame so they don't satisfy the definition of subgame perfection.

4. *Find all the subgame perfect Nash equilibria of the following games. Please give equilibrium strategies as well as payoffs.*

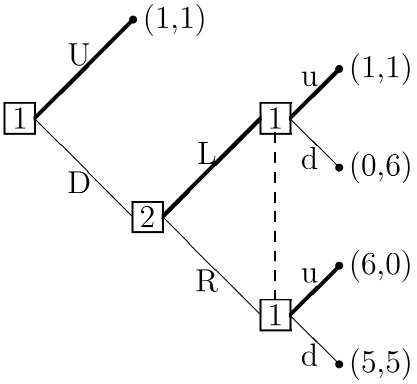
(a)

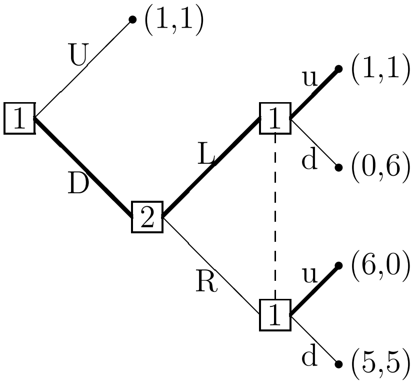


There is only one subgame (other than the entire game). It may be represented in the strategic form as:

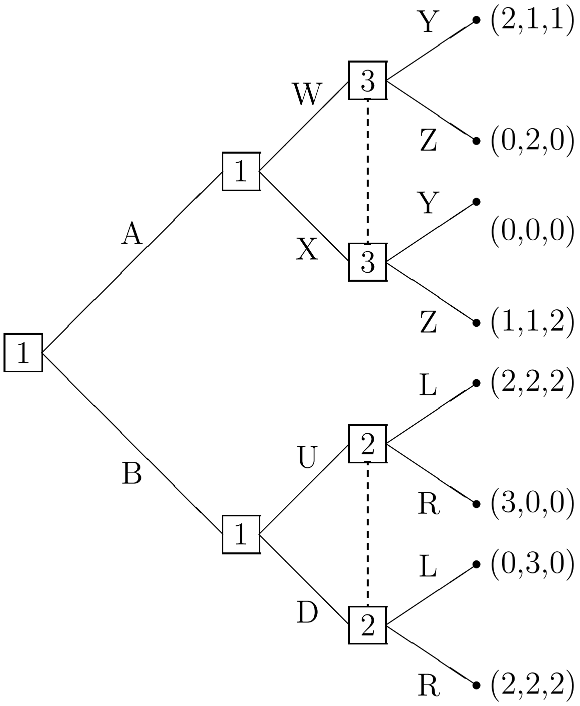
		Player 2	
		L	R
Player 1	u	1,1	6,0
	d	0,6	5,5

It is easy to see that in this game there is just one Nash equilibrium — (u,L) with payoffs (1,1). In the first stage player 1 chooses between playing U (which will pay 1) and playing D which will lead to the subgame discussed above, and (in a subgame perfect Nash equilibrium) will also pay 1. So he is indifferent between the two and hence each choice leads to the Nash equilibrium. Thus there are two subgame perfect Nash equilibria: (Uu, L) and (Du,L) with payoffs of (1,1) each.





(b)

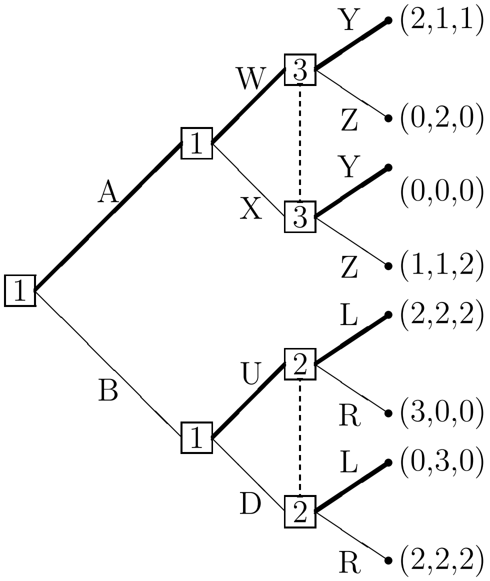


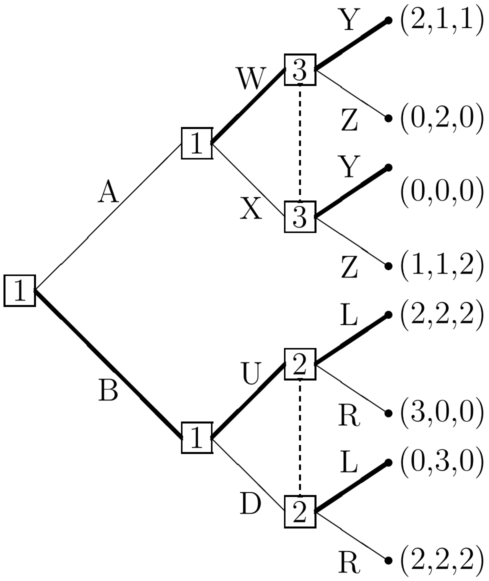
Let's begin with finding Nash equilibria in all of the subgames. In the upper subgame (following player 1's move A) there are two Nash equilibria: (W,Y) and (X,Z). In the lower one, there is just one equilibrium - (U,L). In a subgame perfect equilibrium players' moves form a Nash equilibrium in every subgame. Therefore we need to consider two cases:

Case 1 ((W,Y) played in the upper subgame and (U,L) played in the lower subgame): In this case Player 1 will get a payoff of 2 whether he (or she) plays A or B. Therefore both are Nash equilibria and hence we have the following 2 subgame perfect Nash equilibria for this case:

- (AWU,L,Y) with a payoff (2,1,1)

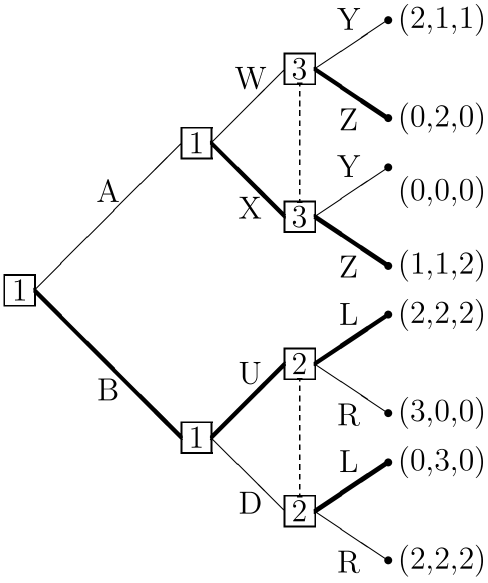
- (BWU,L,Y) with a payoff (2,2,2)





Case 2 ((X,Z) played in the upper subgame and (U,L) played in the lower subgame): In this case Player 1 will get a payoff of 1 by choosing A and a payoff of 2 by choosing B. Therefore the unique Nash equilibrium in this case is B and we have only one subgame perfect Nash equilibrium for this case. It is

- (BXU,L,Z) with a payoff (2,2,2)



EC 308 Question Set # 3

1. Find the mixed strategy Nash equilibrium of the following games.

   (a)

		Player 2	
		L	R
Player 1	U	3,4	9,2
	D	5,1	6,7

   (b)

		Player 2	
		L	R
Player 1	U	3,5	1,2
	D	2,1	6,4

   (c)

		Player 2		
		L	M	R
	u	1,2	4,3	3,2
Player 1	m	0,3	6,2	5,1
	d	0,2	3,1	2,4

2. You are a member of a three person committee that must choose one outcome from the list (A,B,C,D). Suppose the rankings are:

You	Voter 2	Voter 3
A	D	C
B	C	B
D	A	D
C	B	A

   Which of the following procedures would you prefer to see implemented if everyone is sophisticated? (Please show your work.)

   (a) First A vs. B, next C vs. D, and finally the two winners vs. each other.
   (b) First A vs. B, next the winner vs. C, and finally the survivor vs. D.
   (c) First A vs. D, next B vs. C, and finally the two winners vs. each other.
   (d) First D vs. C, next the winner vs. B, and finally the survivor vs. A.

   (Note that at each vote the winner is determined by the majority rule.)

3. (a) Consider the following voting problem: There are four alternatives A, B, C, D, and five voters with the following rankings:

Voter 1	Voter 2	Voter 3	Voter 4	Voter 5
A	A	C	D	C
B	B	D	C	B
C	D	B	B	D
D	C	A	A	A

The order of the voting is such that, first A competes against B, next C competes against D, and finally the two winners compete to determine the final outcome. (At each step the winner is determined by the majority rule.) Find the final outcome if voters 1 and 2 are sophisticated voters and the rest of the voters are sincere voters. Clearly indicate the vote of each voter in the first two round of votes (i.e. the vote between A/B and the two votes between C/D).

(b) Consider the following voting problem: There are five alternatives A, B, C, D, E and five voters with the following rankings:

Voter 1	Voter 2	Voter 3	Voter 4	Voter 5
E	D	B	D	E
A	C	D	B	C
B	B	A	A	D
C	A	C	E	A
D	E	E	C	B

The voting procedure is as follows:

- Round 1: A vs. B
- Round 2:
  - if A is the winner of Round 1 then A vs. C
  - if B is the winner of Round 1 then B vs. D
- Round 3: the winner of Round 2 vs. E

Find the final outcome if voters 1, 2 and 3 are sophisticated voters and the rest of the voters are sincere voters. Clearly indicate the vote of each voter in the first two round of votes (i.e. the votes between A/B, A/C, and B/D).

4. The success of a project depends on the effort level of a worker. In case of success the revenues will be $400,000 whereas in case of no-success the revenues will be $0. The worker can provide a low effort level or a high effort level. The effort level cannot be observed by the principal. The worker requires an expected salary of $60,000 to provide the low effort level and an expected salary of $80,000 to provide the high effort level. The success probabilities for the project are 40% in case of low effort level and 80% in case of high effort level. What should be the minimum bonus (that is awarded in addition to the base salary in case of success) and the base salary for the worker so that the worker has the incentives to provide the high effort level?

Question Set # 3 - Answer Key

1. Remember that a player should be indifferent between the strategies he/she is mixing.

   (a)

			Player 2	
			q	(1-q)
			L	R
Player 1	p	U	3,4	9,2
	(1-p)	D	5,1	6,7

   Both players should be indifferent between their two strategies:

   Player 1: $E(U) = E(D) \Rightarrow 3q + 9 \times (1 - q) = 5q + 6 \times (1 - q) \Rightarrow 5q = 3 \Rightarrow q = 3/5$,
   Player 2: $E(L) = E(R) \Rightarrow 4p + (1-p) = 2p + 7 \times (1-p) \Rightarrow 8p = 6 \Rightarrow p = 3/4$.

   Therefore Player 1 plays U with 3/4 probability and D with 1/4 probability (or simply 3/4U+1/4D), and Player 2 plays L with 3/5 probability and R with 2/5 probability (3/5L+2/5R) at mixed strategy Nash equilibrium.

   (b)

			Player 2	
			q	(1-q)
			L	R
Player 1	p	U	3,5	1,2
	(1-p)	D	2,1	6,4

   Both players should be indifferent between their two strategies:

   Player 1: $E(U) = E(D) \Rightarrow 3q + (1-q) = 2q + 6 \times (1-q) \Rightarrow 6q = 5 \Rightarrow q = 5/6$,
   Player 2: $E(L) = E(R) \Rightarrow 5p + (1-p) = 2p + 4 \times (1-p) \Rightarrow 6p = 3 \Rightarrow p = 1/2$.

   Therefore Player 1 plays (1/2U+1/2D), and Player 2 plays (5/6L+1/6R) at mixed strategy Nash equilibrium.

   (c) Recall that only the strategies that survive iterated elimination of dominated strategies can be played at a mixed strategy Nash equilibrium.

			Player 2	
		L	M	R
	u	1,2	4,3	3,2
Player 1	m	0,3	6,2	5,1
	d	0,2	3,1	2,4

In this game strategy d is dominated for Player 1 and once it is eliminated strategy R is dominated for Player 2. Now we can easily find the mixed strategy Nash equilibrium:

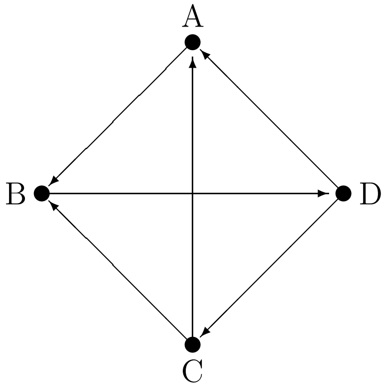
			Player 2	
			q	(1-q)
			L	M
Player 1	p	u	1,2	4,3
	(1-p)	m	0,3	6,2

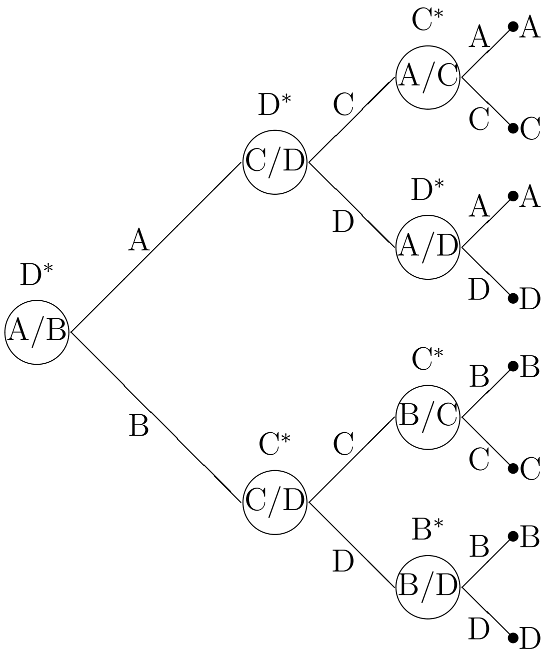
Both players should be indifferent between two remaining strategies:

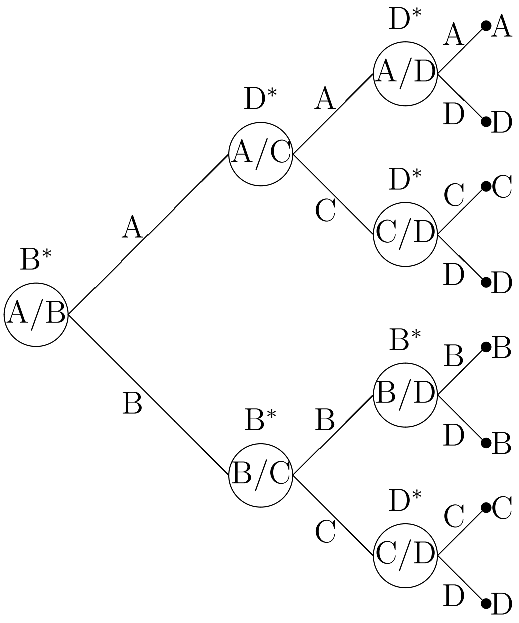
Player 1: $E(u) = E(m) \Rightarrow q + 4 \times (1-q) = 6 \times (1-q) \Rightarrow 3q = 2 \Rightarrow q = 2/3$,
Player 2: $E(L) = E(M) \Rightarrow 2p + 3 \times (1-p) = 3p + 2 \times (1-p) \Rightarrow 2p = 1 \Rightarrow p = 1/2$.

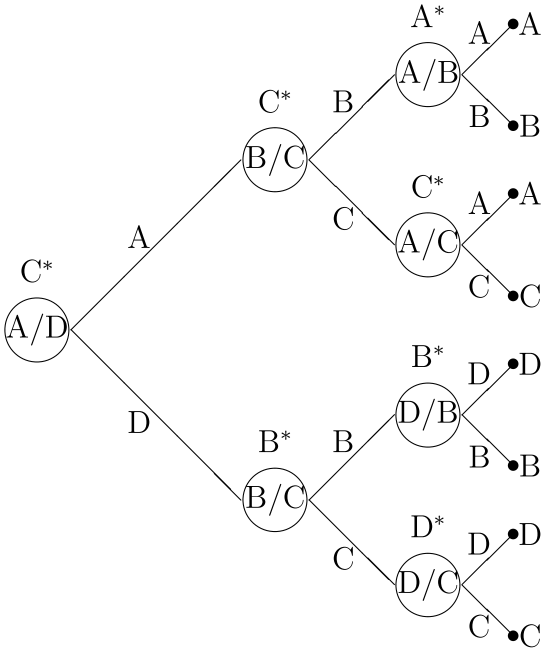
Therefore Player 1 plays (1/2u+1/2m), and Player 2 plays (2/3L+1/3M) at mixed strategy Nash equilibrium.

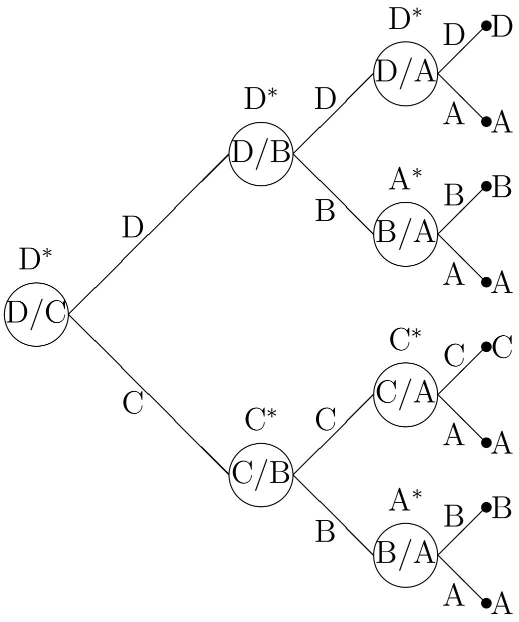
2. Let's first find the majority tournament and use it to find the sophisticated voting outcome of each voting procedure.







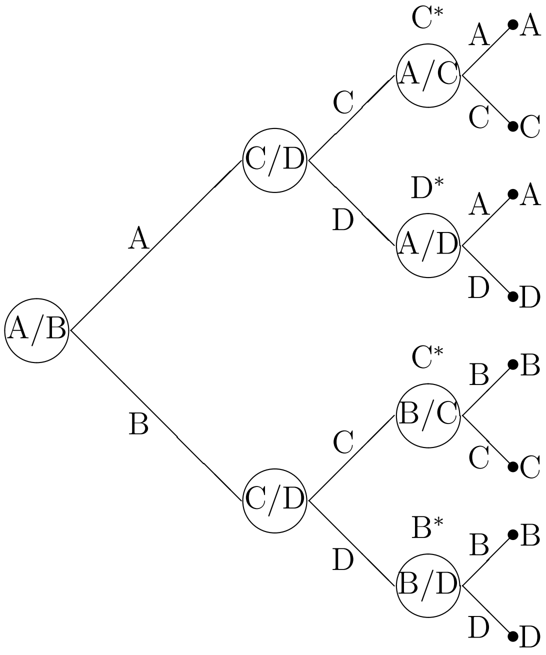




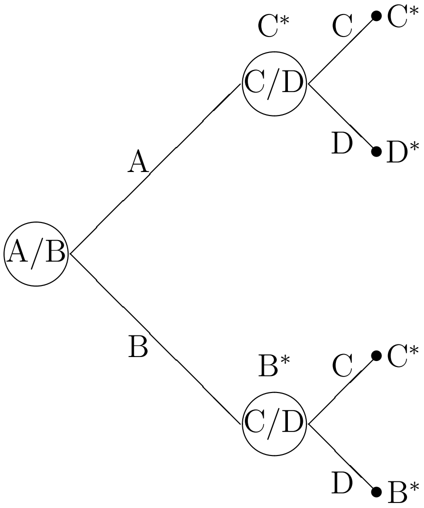
The sophisticated voting outcome is B in voting procedure b, C in voting procedure c, and D in voting procedures a and d. Therefore you would prefer voting procedure b.

3.

(a) Let's start from the last round and move backwards to find the final outcome.
Last Round:



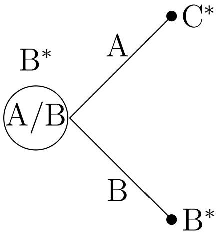
In the last round both sincere and sophisticated agents vote sincerely. Therefore the winner of the vote A vs. C is the alternative C, the winner of the vote A vs. D is the alternative D, the winner of the vote B vs. C is the alternative C, and the winner of the vote B vs. D is the alternative B.
Round 2:



Vote C/D (Top): Everybody views this vote as it is; C wins the vote and moreover this vote yields C as the eventual winner.

Vote C/D (Bottom): Voters 1,2 view this vote as C/B whereas voters 3,4,5 view it as C/D. Voters 1 and 2 both prefer B to C and therefore they vote for D. Voters 3 and 5 both prefer C to D and therefore they vote for C. Voter 4 prefers D to C and therefore votes for D. D wins this vote and this vote eventually yields B as the eventual winner.

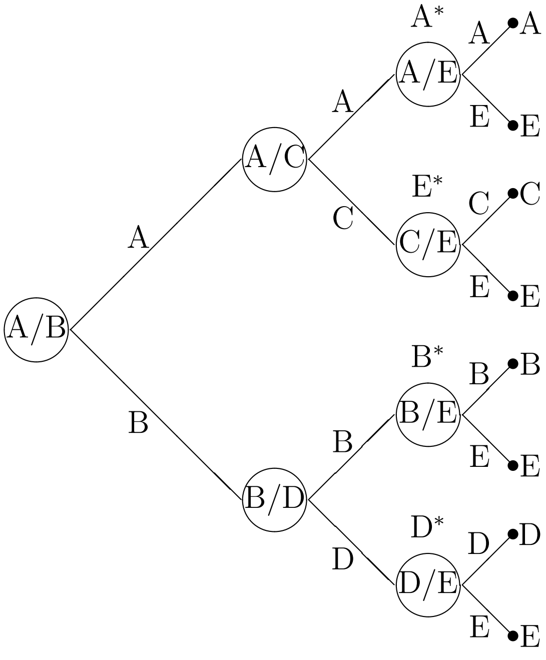
Round 1:



Vote A/B: Voters 1,2 view this vote as C/B whereas voters 3,4,5 view it as A/B. Voters 1 and 2 prefer B to C and therefore vote for B. Voters 3, 4, and 5 all prefer B to A and therefore they vote for B. Alternative B wins this vote and furthermore it is the eventual winner.

(b) Let's start from the last round and move backwards to find the final outcome.

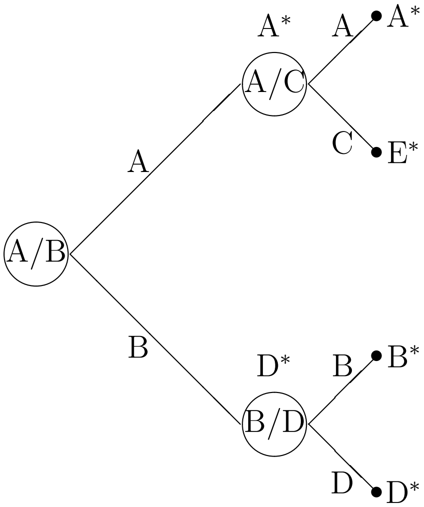
Last Round:



In the last round both sincere and sophisticated agents vote sincerely. Therefore the winner of the vote A vs. E is the alternative A, the winner of the vote

C vs. E is the alternative E, the winner of the vote B vs. E is the alternative B, and the winner of the vote D vs. E is the alternative D.

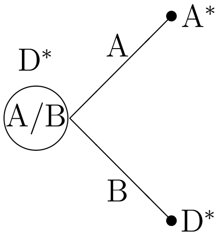
Round 2:



Vote A/C: Voters 1,2, and 3 view this vote as A/E whereas voters 4 and 5 view it as A/C. Voter 1 prefers E to A and therefore votes for C. Voters 2 and 3 both prefer A to E and therefore they vote for A. Voter 4 prefers A to C and therefore votes for A. Voter 5 prefers C to A and therefore votes for C. A wins this vote and this vote eventually yields A as the eventual winner.

Vote B/D: Everybody views this vote as it is; D wins the vote (voters 2, 4, and 5 vote for it) and moreover this vote yields D as the eventual winner.

Round 1:



Vote A/B: Voters 1,2, and 3 view this vote as A/D whereas voters 4 and 5 view it as A/B. Voter 1 prefers A to D and therefore votes for A. Voters 2 and 3 prefer D to A and therefore vote for B. Voter 4 prefers B to A and votes for B. Voter 5 prefers A to B and votes for A. Alternative B wins this vote and it yields alternative D as the eventual winner.

4. The optimal contract is in the following form:

$$\text{Salary} \; = \begin{cases} x & \text{in case the project is not succesful} \\ x+y & \text{in case the project is succesful} \end{cases}$$

We should find x and y. If the worker provides the low effort level his/her expected salary is 0.4(x+y) + 0.6x = x+0.4y. If the worker provides the high effort level his/her expected salary is 0.8(x+y) + 0.2x = x+0.8y. Therefore the difference between his/her expected salaries from high and low effort is x+0.8y - (x+0.4y) = 0.4y. On the other hand the difference the worker requires to provide the high effort level is 80,000-60,000 = 20,000. Therefore

$$0.4y = 20,000 \Rightarrow y = 50,000.$$

The bonus should be $50,000. Next let's find the base salary? Recall that the expected salary for a high effort worker is $80,000. Therefore

$$0.8(x + 50,000) + 0.2x = 80,000 \Rightarrow x + 40,000 = 80,000 \Rightarrow x = 40,000.$$

The base salary should be $40,000. Therefore the optimal contract for the firm is as follows:

$$\text{Salary } = \begin{cases} 40,000 & \text{in case the project is not successful} \\ 90,000 & \text{in case the project is successful} \end{cases}$$

EC 308 Question Set # 4

1. After an accident three identical kidneys became available for transplantation as a result of the death of identical twins. (One of the kidneys was damaged.)
    (a) Consider 4 patients who need kidney transplantation and who differ only in the time they waited for transplantation and the number of antigen matches they have with the available kidneys: Patient A waited for 4 years and has 3 antigen matches; patient B waited for 3 years and has 2 antigen matches; patient C waited for 2 years and has 2 antigen matches; and patient D waited for 1 year and has 4 antigen matches. Which patients receive the kidneys? Explain.
    (b) Consider an alternative scenario: Suppose one of the twins died immediately and two kidneys became available. Who receives the kidneys among A, B, C, D? After a month (and after two kidney operations are carried out) the other twin who was in coma and who had a damaged kidney died and one more kidney became available. Which patient receives the last kidney?
    (c) Does the results in a and b make sense? Why or why not?

2. There are three alternatives A, B, C and 100 voters with the following rankings:

19 voters	16 voters	20 voters	10 voters	13 voters	22 voters
A	A	B	B	C	C
B	C	C	A	B	A
C	B	A	C	A	B

    (a) Is there a majority alternative? Why or why not?
    (b) Is there a paradox of voting here? Explain.
    (c) What is the Borda Score of each alternative?
    (d) What is the Condorcet score of each ranking?

3. Consider the following apportionment problem: There are a total of 16 seats to be allocated. There are four states with the following populations: State A (150), State B (190), State C (270), and State D (390).
    (a) Find the allocations suggested by Hamilton's, Jefferson's, Adam's, and Hill's methods. (Please show your work.)
    (b) Try to do the same for the Webster's method. What is the problem? (In general such problems may happen when the numbers are very round.)
    (c) Repeat the problem for the case of a total of 17 seats to be allocated. Is there an example of a paradox here? Please explain.

Question Set # 4 - Answer Key

1. (a) The patients get points for number of antigens matched (2 for each) and for the time that they waited. The points each of them gets are:

Patient	Point for waiting	Points for antigens matched	Total
A	10	6	16
B	7.5	4	11.5
C	5	4	9
D	2.5	8	10.5

   When three kidneys became available they should be allocated to individuals with the highest number of points i.e. A, B and D.

   (b) With two kidneys available they will be assigned to A and D. Now we have to reevaluate scores of the individuals who are still on the waiting list:

Patient	Point for waiting	Points for antigens matched	Total
C	10	4	14
D	5	8	13

   When the additional kidney becomes available it should be assigned to C since this is the patient with the highest number of points now.

   (c) The rule is not consistent. Even though in both scenarios the same number of kidneys is assigned, different individuals receive them.

2. Let's start with summarizing the outcomes of voting on all possible pairs of alternatives (i.e. number of votes each of them gets):

	A	B	C
A against B	57	43	—
A against C	45	—	55
B against C	—	49	51

   (a) C beats both alternatives so it is the majority alternative.

   (b) There is no paradox voting because there is no cycle in majority voting.

   (c) Borda scores (each column contains the number of points assigned by the particular type of voters to a given alternative):

	19 voters	16 voters	20 voters	10 voters	13 voters	22 voters	score
A	2	2	0	1	0	1	102
B	1	0	2	2	1	0	92
C	0	1	1	0	2	2	106

The Borda ranking is CAB

(d) Condorcet scores are:

Ranking	Individuals who support:			Score
	1^{st} vs 2^{nd}	1^{st} vs 3^{rd}	2^{nd} vs 3^{rd}	
ABC	57	45	49	151
ACB	45	57	51	153
BAC	43	49	45	137
BCA	49	43	55	147
CAB	55	51	57	163
CBA	51	55	43	149

The Condorcet ranking is CAB

3. The total population is 1000.

(a) **Hamilton's method:** to get each state's quota we have to multiply its population by 16 and divide by 1000 (equivalently divide it by $\frac{1000}{16} = 62.5$):

$$\begin{array}{rl} \text{A:} & \frac{150}{62.5} = 2.40 \\ \text{B:} & \frac{190}{62.5} = 3.04 \\ \text{C:} & \frac{270}{62.5} = 4.32 \\ \text{D:} & \frac{390}{62.5} = 6.24 \end{array}$$

First, state A gets 2 seats, state B gets 3 seats, state C gets 4 seats and state D gets 6 seats. 15 seats are assigned, the remaining one goes to A.

Jefferson's method: initial divisor is 62.5, and the quotients are as in the Hamilton's method. The total of 15 seats could be allocated in this way so the divisor should be decreased. The highest number that works is 55.714, but 55 gives also the same answer:

$$\begin{array}{rl} \text{A:} & \frac{150}{55} = 2.73 \\ \text{B:} & \frac{190}{55} = 3.45 \\ \text{C:} & \frac{270}{55} = 4.91 \\ \text{D:} & \frac{390}{55} = 7.09 \end{array}$$

State A gets 2 seats, state B gets 3 seats, state C gets 4 seats and state D gets 7 seats.

Adams's method: using 62.5 as the common divisor would allocate too many (19) seats so we have to increase a divisor. The smallest number that allows are to assign the seats is 67.5:

$$\begin{array}{rl} \text{A:} & \frac{150}{67.5} = 2.22 \\ \text{B:} & \frac{190}{67.5} = 2.81 \\ \text{C:} & \frac{270}{67.5} = 4.00 \\ \text{D:} & \frac{390}{67.5} = 5.77 \end{array}$$

3 seats go to the state A, 3 to the state B, C gets 4 seats and D — 6.

Hill's method: With the initial divisor:

$$\begin{aligned} &\text{A:} \quad \tfrac{150}{62.5} = 2.40 < \sqrt{2 \times 3} = 2.45 \\ &\text{B:} \quad \tfrac{190}{62.5} = 3.04 < \sqrt{3 \times 4} = 3.46 \\ &\text{C:} \quad \tfrac{270}{62.5} = 4.32 < \sqrt{4 \times 5} = 4.47 \\ &\text{D:} \quad \tfrac{390}{62.5} = 6.24 < \sqrt{6 \times 7} = 6.48 \end{aligned}$$

2 seats would be allocated to A, 3 to B, 4 to C and 6 to D — total of 15 seats so we need to decrease the divisor. The biggest one that works is (approximately) 61.24, but 61 also works fine:

$$\begin{aligned} &\text{A:} \quad \tfrac{150}{61} = 2.46 > \sqrt{2 \times 3} = 2.45 \\ &\text{B:} \quad \tfrac{190}{61} = 3.11 < \sqrt{3 \times 4} = 3.46 \\ &\text{C:} \quad \tfrac{270}{61} = 4.43 < \sqrt{4 \times 5} = 4.47 \\ &\text{D:} \quad \tfrac{390}{61} = 6.39 < \sqrt{6 \times 7} = 6.48 \end{aligned}$$

3 seats are allocated to A, 3 to B, 4 to C and 6 to D.

(b) **Webster's method:** it's easy to see that the initial divisor does not work well. Consider divisor of 60:

$$\begin{aligned} &\text{A:} \quad \tfrac{150}{60} = 2.50 \\ &\text{B:} \quad \tfrac{190}{60} = 3.167 \\ &\text{C:} \quad \tfrac{270}{60} = 4.50 \\ &\text{D:} \quad \tfrac{390}{60} = 7.50 \end{aligned}$$

One can notice that as long as the divisor is greater than 60, 15 seats will be allocated. When it is greater than 60 — 18 seats are allocated. When it is exactly 60 we have to decide if we should round up .5 to 1 or to 0. If we decide to round it up to 0, 15 seats will be allocated, when we decide to round it up to 1 — total of 18 seats will be allocated. The problem is that as we change the divisor in order to allocate more seats, a few states "switch" to the higher allotment for precisely the same value of divisor.

(c) **Hamilton's method:** to get each state's quota we have to multiply its population by 17 and divide by 1000 (equivalently divide it by $\frac{1000}{17} = 58.823529$):

$$\begin{aligned} &\text{A:} \quad \tfrac{150}{1000} \times 17 = 2.55 \\ &\text{B:} \quad \tfrac{190}{1000} \times 17 = 3.23 \\ &\text{C:} \quad \tfrac{270}{1000} \times 17 = 4.59 \\ &\text{D:} \quad \tfrac{390}{1000} \times 17 = 6.63 \end{aligned}$$

First 15 seats are assigned identically as in the initial case. Another two go to D and C.

Jefferson's method: the highest number that works is 54:

$$\begin{array}{ll} \text{A:} & \frac{150}{54} = 2.78 \\ \text{B:} & \frac{190}{54} = 3.52 \\ \text{C:} & \frac{270}{54} = 5.00 \\ \text{D:} & \frac{390}{54} = 7.22 \end{array}$$

State A gets 2 seats, state B gets 3 seats, state C gets 5 seats and state D gets 7 seats.

Adams's method: now the smallest number that works is 65

$$\begin{array}{ll} \text{A:} & \frac{150}{65} = 2.31 \\ \text{B:} & \frac{190}{65} = 2.92 \\ \text{C:} & \frac{270}{65} = 4.15 \\ \text{D:} & \frac{390}{65} = 6.00 \end{array}$$

3 seats go to the state A, 3 to the state B, C gets 5 seats and D — 6.

Hill's method: Using common divisor ($\frac{1000}{17} = 58.823529$) we get:

$$\begin{array}{ll} \text{A:} & \frac{150}{1000} \times 17 = 2.55 > \sqrt{2 \times 3} = 2.45 \\ \text{B:} & \frac{190}{1000} \times 17 = 3.23 < \sqrt{3 \times 4} = 3.46 \\ \text{C:} & \frac{270}{1000} \times 17 = 4.59 > \sqrt{4 \times 5} = 4.47 \\ \text{D:} & \frac{390}{1000} \times 17 = 6.63 > \sqrt{6 \times 7} = 6.48 \end{array}$$

Total of 18 seats would be allocated. We have to increase the divisor, the smallest one that works 60.19 (notice that neither 60 nor 61 don't allocate 17 seats):

$$\begin{array}{ll} \text{A:} & \frac{150}{60.19} = 2.49 > \sqrt{2 \times 3} = 2.45 \\ \text{B:} & \frac{190}{60.19} = 3.16 < \sqrt{3 \times 4} = 3.46 \\ \text{C:} & \frac{270}{60.19} = 4.49 > \sqrt{4 \times 5} = 4.47 \\ \text{D:} & \frac{390}{60.19} = 6.479 < \sqrt{6 \times 7} = 6.481 \end{array}$$

3 seats are assigned to state A, 3 to state B, 5 to state C and 6 to state D.

Webster's method: the same problem as before appears also here. We can allocate either 15 or 18 seats but it is indeterminate in between.

Question Set # 5

1. There are three claimants with the following claims: Claim A = \$ 100, Claim B = \$ 150, Claim C = \$ 400. There is a total of \$ 450 to allocate.
   - (a) Find the claims allocation suggested by the proportional rule.
   - (b) Find the claims allocation suggested by the Shapley value.
   - (c) Find the claims allocation suggested by the Talmudic solution.
   - (d) Find the claims allocation suggested by the Maimonides's rule.
2. Consider the following cost sharing game: There are three players with the following costs: c(A) = c(B) = 50, c(C) = 60, c(A,B) = c(A,C) = 80, c(B,C) = 70, and c(A,B,C) = 100.
   - (a) Graphically identify the core of this cost sharing game.
   - (b) Find the Shapley value of this cost sharing game. Is the Shapley value in the core? Why or why not?

Question Set # 5 - Answer Key

1. The sum of claims is 650, amount available is 450.

   (a) The proportional rule: each individual gets the fraction of the total proportional to his/her claim — A gets $\frac{100}{650} \times 450 = 69.23$, B gets $\frac{150}{650} \times 450 = 103.85$ and C gets $\frac{400}{650} \times 450 = 276.92$.

   (b) The Shapley value:

Order	A	B	C
ABC	100	150	200
ACB	100	0	350
BAC	100	150	200
BCA	0	150	300
CAB	50	0	400
CBA	0	50	400
Total	350	500	1850
Average	$58\frac{1}{3}$	$83\frac{1}{3}$	$308\frac{1}{3}$

   Therefore the Shapley value allocation is A: $58\frac{1}{3}$, B: $83\frac{1}{3}$ and C: $308\frac{1}{3}$.

   (c) Talmudic solution: the total amount is greater than the half of the sum of all claims, therefore each individual gets first a half of his/her claim (so total allocated in this way is $\frac{650}{2} = 325$) and we need to allocate the remaining sum (125). At this point the loss of the individual A is 50, the loss of the individual B is 75 and the loss of the individual C is 200. First we have to allocate money to individual C (until his loss is the same as that of individual B) — so 125 goes to C (so that the losses of C and B are both 75). Thus the final allocation is A — 50, B — 75 and C — 325.

   (d) The allocation suggested by the Maimonides rule: A gets 100, B gets 150, C gets 200. (When everyone gets 100, the total is 300; we have 150 more to allocate between B and C. When they get additional 50, the total is 400. The remaing 50 goes to C.)

2. (a) The shaded region in the picture below is the core of the cost sharing game.

(b) The table below presents calculations. The Shapley allocations is A — 35, B — 30 and C — 35. None of the individuals and none of the coalition pays more than it would pay when acting separately so the allocation is in the core.

Order	A	B	C
ABC	50	30	20
ACB	50	20	30
BAC	30	50	20
BCA	30	50	20
CAB	20	20	60
CBA	30	10	60
Total	210	180	210
Average	35	30	35