Mixed Strategies

Consider the matching pennies game:

		Player 2: Heads	Player 2: Tails
Player 1	Heads	1,-1	-1,1
	Tails	-1,1	1,-1

- There is no (pure strategy) Nash equilibrium in this game. If we play this game, we should be "unpredictable." That is, we should randomize (or mix) between strategies so that we do not get exploited.

- But not any randomness will do: Suppose Player 1 plays .75 Heads and .25 Tails (that is, Heads with 75% chance and Tails with 25% chance). Then Player 2 by choosing Tails (with 100% chance) can get an expected payoff of $0.75\times1 + 0.25\times(-1) = 0.5$. But that cannot happen at equilibrium since Player 1 then wants to play Tails (with 100% chance) deviating from the original mixed stategy.

- Since this game is completely symmetric it is easy to see that at **mixed strategy Nash equilibrium** both players will choose Heads with 50% chance and Tails with 50% chance.

- In this case the expected payoff to both players is $0.5\times1 + 0.5\times(-1) = 0$ and neither can do better by deviating to another strategy (regardless it is a mixed strategy or not).

- In general there is no guarantee that mixing will be 50-50 at equilibrium.

Example (Tennis):

		Server F	Server B
Receiver	F	90,10	20,80
	B	30,70	60,40

- Here the payoffs to the Receiver is the probability of saving and the payoffs to the Server is the probability of scoring.
- Let's consider the potential strategies for the Server:
  - If the Server always aims Forehands then the Receiver (anticipating the Forehand serve) will always move Forehands and the payoffs will be (90,10) to Receiver and Server respectively.
  - If the Server always aims Backhands then the Receiver (anticipating the Backhand serve) will always move Backhands and the payoffs will be (60,40).

- How can the Server do better than that? The Server can increase his performance by mixing Forehands and Backhands.
- For example suppose the Server aims Forehand with 50% chance and Backhands with 50% chance (or simply mixes 50-50). Then the Receiver's payoff is

  * $0.5 \times 90 + 0.5 \times 20 = 55$ if she moves Forehands and
  * $0.5 \times 30 + 0.5 \times 60 = 45$ if she moves Backhands.

  Since it is better to move Forehands, she will do that and her payoff will be 55. Therefore if the Server mixes 50-50 his payoff will be 45. (Note that the payoffs add up to 100). This is already an improvement for the Server's performance.

- The next step is searching for the best mix for the Server. How can he get the best performance?
- Suppose the Server aims Forehands with q probability and Backhands with 1-q probability. Then the Receiver's payoff is
  * $q \times 90 + (1\text{-}q) \times 20 = 20 + 70q$ if she moves Forehands and
  * $q \times 30 + (1\text{-}q) \times 60 = 60 - 30q$ if she moves Backhands.
- The Receiver will move towards the side that maximizes her payoff. Therefore she will move
  - Forehands if $20 + 70q > 60 - 30q$,
  - Backhands if $20 + 70q < 60 - 30q$, and
  - either one if $20 + 70q = 60 - 30q$.

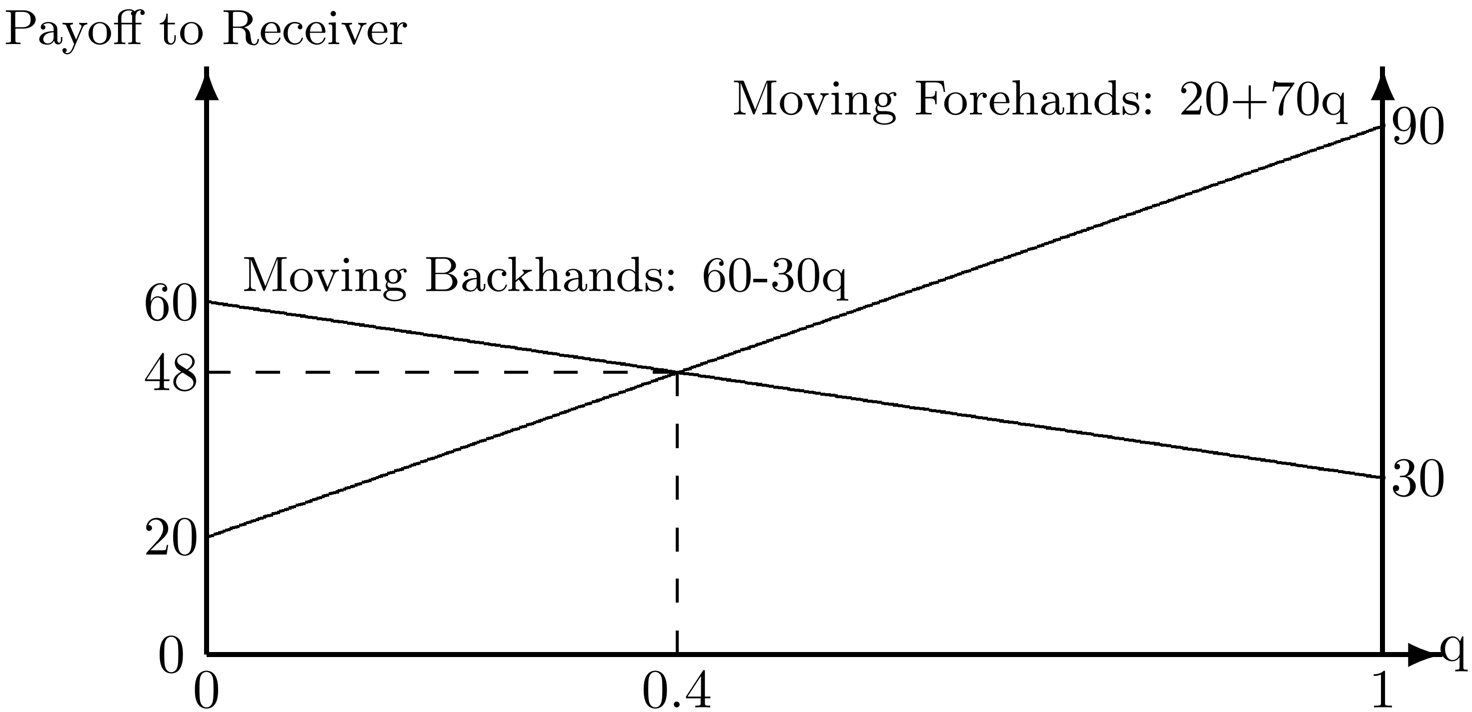
  That is the Receiver's payoff is the larger of $20+70q$ and $60-30q$.

- The server, to maximize his payoff, should minimize the Receiver's payoff. He can do that by setting 20+70q and 60-30q equal:

$$20 + 70q = 60 - 30q \Rightarrow 100q = 40 \Rightarrow q = 0.4.$$

- In order to maximize his payoff the Server should aim Forehands 40% of the time and Backhands 60% of the time. In this case the Receiver's payoff will be $20 + 70\times0.4 = 60$ - $30\times0.4 = 48$.

- In other words if the Server mixes 40-60 then the Receiver's payoff will be 48 whether she moves Forehands or Backhands (or mixes between them). Therefore the Server's payoff will be 100-48 $= 52$.



Payoff to Receiver
Moving Forehands: 20+70q
90
Moving Backhands: 60-30q
60
48
20
30
0
0
0.4
1
q

Next let's carry out a similar analysis for the Receiver.

- If the Receiver does not mix, then the Server will aim the other side.
- Suppose the Receiver moves Forehands with p probability. Then her payoff is
  * $p \times 90 + (1-p) \times 30 = 30 + 60p$ if the Server aims Forehands and
  * $p \times 20 + (1-p) \times 60 = 60 - 40p$ if the Server aims Backhands.
- The Server will aim towards the side that minimizes the Receiver's payoff. Therefore he will aim
  - Forehands if $30 + 60p < 60 - 40p$,
  - Backhands if $30 + 60p > 60 - 40p$, and
  - either one if $30 + 60p = 60 - 40p$.

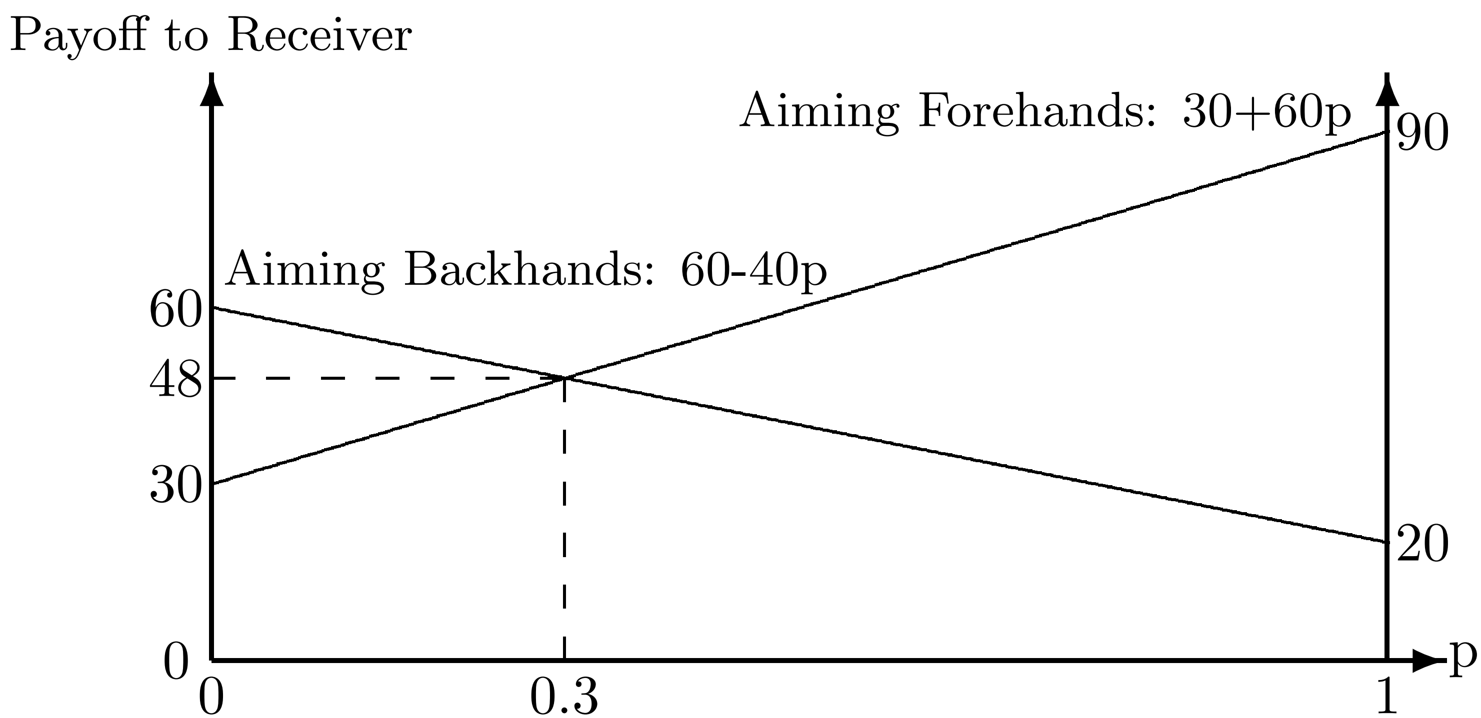
- That is, the Receiver's payoff is the smaller of 30+60p and 60-40p. The Receiver should equate 30+60p and 60-40p so as to maximize her payoff:

$$30 + 60p = 60 - 40p \Rightarrow 100p = 30 \Rightarrow p = 0.3.$$

- In order to maximize her payoff the Receiver should move Forehands 30% of the time and Backhands 70% of the time. In this case the Receiver's her payoff will be $30 + 60 \times 0.3 = 60$ - $40 \times 0.3 = 48$. Therefore the Server's payoff will be 100-48 = 52.

- Therefore the mixed strategy:
  - Receiver: 0.3F + 0.7B, and
  - Server: 0.4F + 0.6B

  is the only one that cannot be "exploited" by either player. Hence it is a mixed strategy Nash equilibrium.


Payoff to Receiver
Aiming Forehands: 30+60p
Aiming Backhands: 60-40p
90
60
48
30
20
0
0
0.3
1
p

Important Observation: If a player is using a mixed strategy at equilibrium, then he/she should have the same expected payoff from the strategies he/she is mixing. We can easily find the mixed strategy Nash equilibrium in 2×2 games using this observation.

Example: Let's find the mixed strategy Nash equilibrium of the following game which has no pure strategy Nash equilibrium.

			Player 2	
			q	(1-q)
			L	R
Player 1	p	U	2,-3	1,2
	(1-p)	D	1,1	4,-1

Let p be the probability of Player 1 playing U and q be the probability of Player 2 playing L at mixed strategy Nash equilibrium. Our objective is finding p and q.

- At mixed strategy Nash equilibrium both players should have same expected payoffs from their two strategies.

- Consider Player 1.
  - If she plays U she'll receive a payoff of 2 with probability q and 1 with probability (1-q). Therefore her expected payoff E(U) from playing U is 2q+(1-q).
  - If she plays D she'll receive a payoff of 1 with probability q and 4 with probability (1-q). Therefore her expected payoff E(D) from playing D is q+4(1-q).

  She'll mix between the two strategies only if these two expected payoffs are the same:

  $$E(U) = E(D) \Rightarrow 2q+(1-q) = q+4(1-q) \Rightarrow 4q = 3 \Rightarrow q = 3/4.$$

  Therefore Player 1 will mix between the two strategies only if q=3/4.

- Next let's consider Player 2.
  - If she plays L she'll receive a payoff of -3 with probability p and 1 with probability (1-p). Therefore her expected payoff E(L) from playing L is -3p+(1-p).
  - If she plays R she'll receive a payoff of 2 with probability p and -1 with probability (1-p). Therefore her expected payoff E(R) from playing R is 2p+(-1)(1-p).

  She'll mix between the two strategies only if these two expected payoffs are same:

  $$E(L) = E(R) \Rightarrow -3p+(1-p) = 2p-(1-p) \Rightarrow 7p = 2 \Rightarrow p = 2/7.$$

  Therefore Player 2 will mix between the two strategies only if p=2/7.

- Therefore the mixed strategy Nash equilibrium is:
  - Player 1: U with probability 2/7 and D with probability 5/7,
  - Player 2: L with probability 3/4 and R with probability 1/4.
- What about the mixed Nash equilibrium payoffs? The payoff for Player 1 is

$$\left(2 \times \frac{3}{4}\right) + \left(1 \times \frac{1}{4}\right) = \left(1 \times \frac{3}{4}\right) + \left(4 \times \frac{1}{4}\right) = \frac{7}{4}$$

  and the mixed Nash equilibrium payoff to Player 2 is

$$\left(-3 \times \frac{2}{7}\right) + \left(1 \times \frac{5}{7}\right) = \left(2 \times \frac{2}{7}\right) + \left(-1 \times \frac{5}{7}\right) = -\frac{1}{7}.$$

Example: There can be mixed strategy Nash equilibrium even if there are pure strategy Nash equilibria.

			Player 2	
			q	(1-q)
			L	R
Player 1	p	U	3,1	0,0
	(1-p)	D	0,0	1,3

At the mixed Nash equilibrium Both players should be indifferent between their two strategies:

- Player 1: $E(U) = E(D) \Rightarrow 3q = 1 - q \Rightarrow 4q = 1 \Rightarrow q = 1/4$,
- Player 2: $E(L) = E(R) \Rightarrow p = 3 \times (1 - p) \Rightarrow 4p = 3 \Rightarrow p = 3/4$.

Therefore Player 1 plays (3/4U+1/4D) and Player 2 plays (1/4L+3/4R) at mixed strategy Nash equilibrium.

Hints for Finding the Mixed Nash Equilibria in Larger Games

- Dominated strategies are never used in mixed Nash equilibria, even if they are dominated by another mixed strategy.

  For example in the following game strategy M is dominated by the mixed strategy (0.5U+0.5D) and therefore Player 1 can mix between only U and D.

  Player 2

Player 1		L	R
	U	3,1	0,2
	M	1,2	1,1
	D	0,4	3,1

  In other words finding its mixed strategy Nash equilibria is equivalent to finding the mixed Nash equilibria of the following game:

		Player 2	
		L	R
Player 1	U	3,1	0,2
	D	0,4	3,1

- Indeed only the strategies that survive iterated elimination of dominated strategies can be used in mixed Nash equilibria.

  Example: In the following game M is dominated by U for Player 1 and next m is dominated by l for Player 2:

		Player 2		
		l	m	r
	U	3,2	2,1	1,3
Player 1	M	2,1	1,5	0,3
	D	1,3	4,2	2,2

Therefore we can find its mixed Nash equilibria by simply finding the mixed Nash equilibria of the following 2 by 2 game:

		Player 2	
		l	r
Player 1	U	3,2	1,3
	D	1,3	2,2