Finding Strategic Game Equivalent of an Extensive Form Game

- In an extensive form game, a strategy for a player should specify what action the player will choose at each information set. That is, a strategy is a complete plan for playing a game for a particular player.
- Therefore to find the **strategic game equivalent** of an extensive form game we should follow these steps:

1. First we need to find all strategies for every player. To do that first we find all information sets for every player (including the singleton information sets). If there are many of them we may label them (such as Player 1's 1st info. set, Player 1's 2nd info. set, etc.)

2. A strategy should specify what action the player will take in *every* information set which belongs to him/her. We find all combinations of actions the player can take at these information sets.

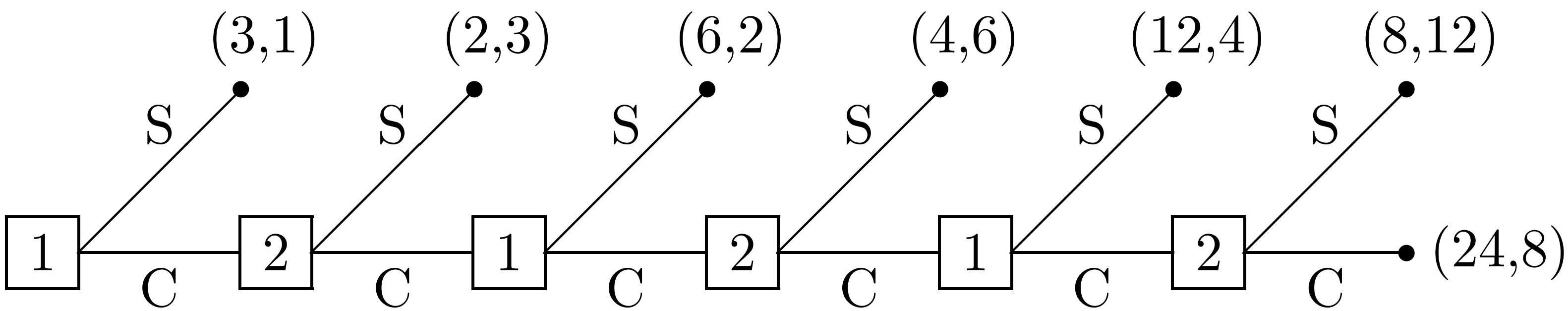
   For example if a player has 3 information sets where

   - the first one has 2 actions,
   - the second one has 2 actions, and
   - the third one has 3 actions

   then there will be a total of $2 \times 2 \times 3 = 12$ strategies for this player.

3. Once the strategies are obtained for every player the next step is finding the payoffs. For every strategy profile we find the payoff vector that would be obtained in case these strategies are played in the extensive form game.

Example: Let's find the strategic game equivalent of the following 3-level centipede game.



1. Both players have 3 information sets.
2. For both players each information set has 2 actions. Therefore both players have $2 \times 2 \times 2 = 8$ strategies.

   Player 1's Strategies: CCC, CCS, CSC, CSS, SCC, SCS, SSC, SSS.

   Player 2's Strategies: CCC, CCS, CSC, CSS, SCC, SCS, SSC, SSS.

3. The last step is finding the payoffs for each of the $8 \times 8 = 64$ cells! This is much easier than it looks:

   - Suppose Player 1's strategy is one of SCC, SCS, SSC or SSS. Since the first component is S, Player 1 Shirks immediately. Hence nomatter what his intentions are for the later information sets and nomatter what Player 2 intends to do, the payoff vector is (3,1).
   - Suppose Player 1's strategy is one of CCC, CCS, CSC or CSS and Player 2's strategy is one of SCC, SCS, SSC or SSS. Then Player 1 Cooperates and next Player 2 Shirks and the payoff vector is (2,3).

   In this way we can fill all the slots.

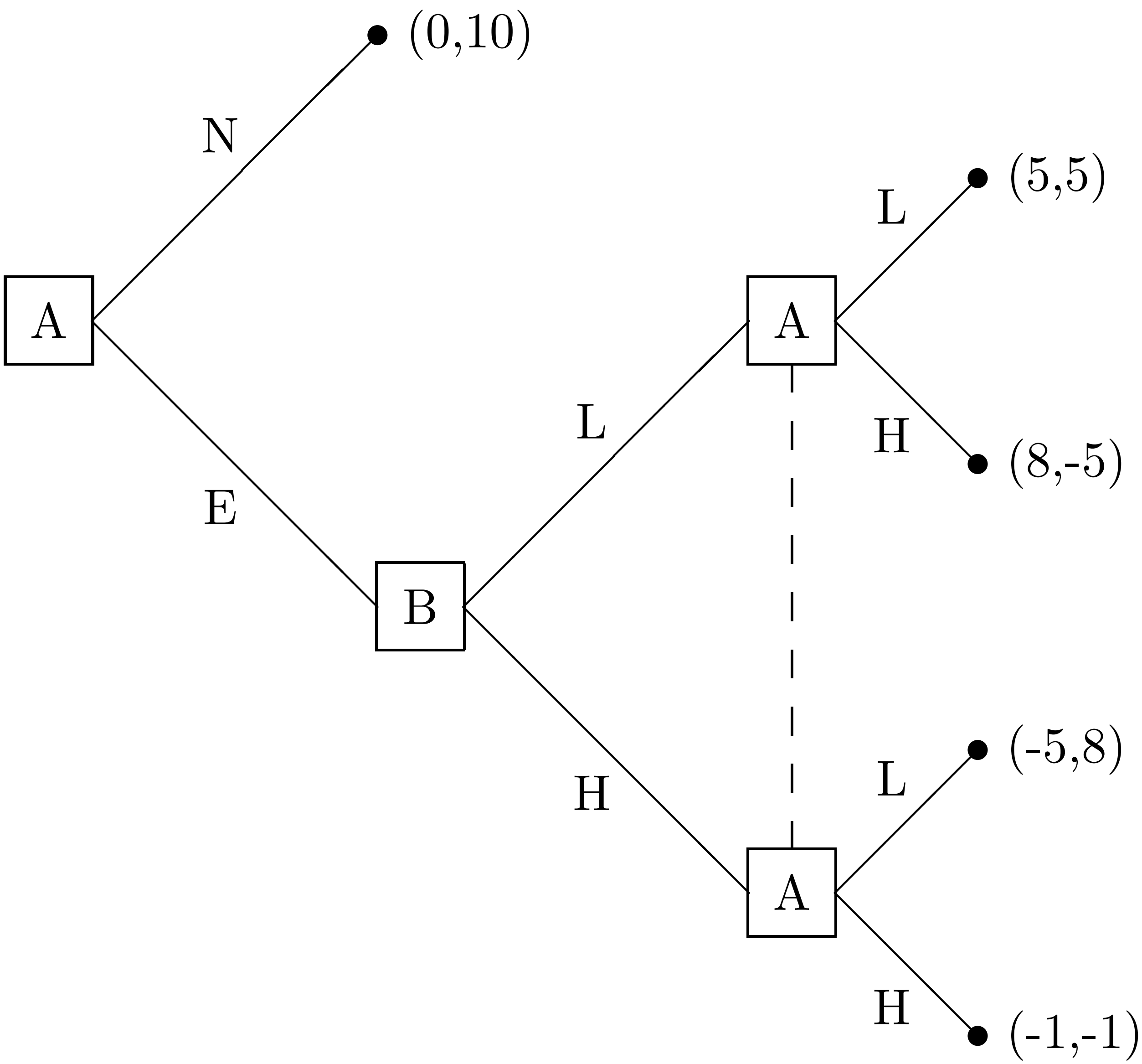
P2

P1		CCC	CCS	CSC	CSS	SCC	SCS	SSC	SSS
	CCC	24,8	8,12	4,6	4,6	2,3	2,3	2,3	2,3
	CCS	12,4	12,4	4,6	4,6	2,3	2,3	2,3	2,3
	CSC	6,2	6,2	6,2	6,2	2,3	2,3	2,3	2,3
	CSS	6,2	6,2	6,2	6,2	2,3	2,3	2,3	2,3
	SCC	3,1	3,1	3,1	3,1	3,1	3,1	3,1	3,1
	SCS	3,1	3,1	3,1	3,1	3,1	3,1	3,1	3,1
	SSC	3,1	3,1	3,1	3,1	3,1	3,1	3,1	3,1
	SSS	3,1	3,1	3,1	3,1	3,1	3,1	3,1	3,1

Example: Firm B is currently a monopolist in a market and Firm A is considering entry. If Firm A does not enter (action N) then the payoffs are (0,10). If it does (action E) they play the following simultaneous game:

		Firm B	
		L	H
Firm A	L	5,5	-5,8
	H	8,-5	-1,-1

This whole game is neither a simultaneous game, nor a sequential game. It is a hybrit with the following extensive form:


A
N
(0,10)
E
B
L
H
A
L
(5,5)
H
(8,-5)
A
L
(-5,8)
H
(-1,-1)

Let's find the strategic game equivalent of this extensive form game.

- Here A has 2 information sets, each with 2 actions.

  A's strategies are: NL, NH, EL, and EH.

- B has only one information set with 2 actions.

  B's strategies are: L and H.

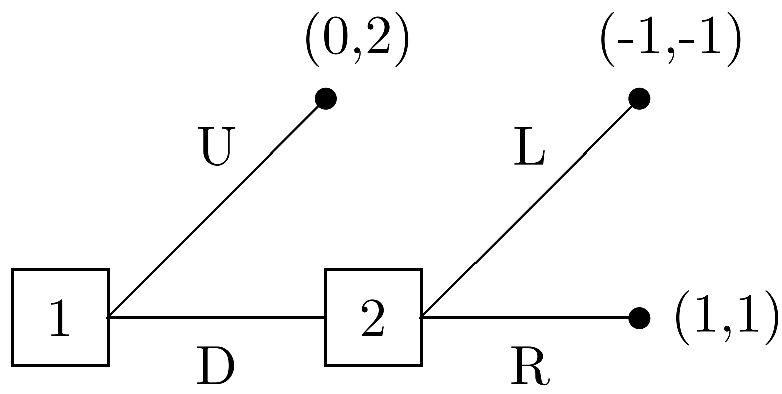
- Now we can easily construct the equivalent matrix game:

		Firm B	
		L	H
	NL	0,10	0,10
Firm A	NH	0,10	0,10
	EL	5,5	-5,8
	EH	8,-5	-1,-1

Nash Equilibrium of Extensive Form Games

- The definition of Nash equilibrium is same for the extensive form games: A strategy profile is a **Nash equilibrium** if no player wants to deviate to another strategy.
- To find the Nash equilibrium of extensive form games the easiest way is first finding the equivalent strategic form game and next finding the Nash equilibrium of this strategic form game.
- For instance in the above example there are two Nash equilibria: (NH,H) and (NL,H).
- However the second Nash equilibrium is "unreasonable." Here Firm A is playing NL and Firm B is playing H. But if Firm A ever enters, given that B is playing H why would it play L? The next example reinforces this point.

Example:



In this game both players have one information set and the matrix equivalent of this game is:

Player 2

		L	R
Player 1	U	0,2	0,2
	D	-1,-1	1,1

- The two Nash equilibria of this game are (D,R) and (U,L).
- The second of these Nash equilibria (U,L) is "unreasonable."

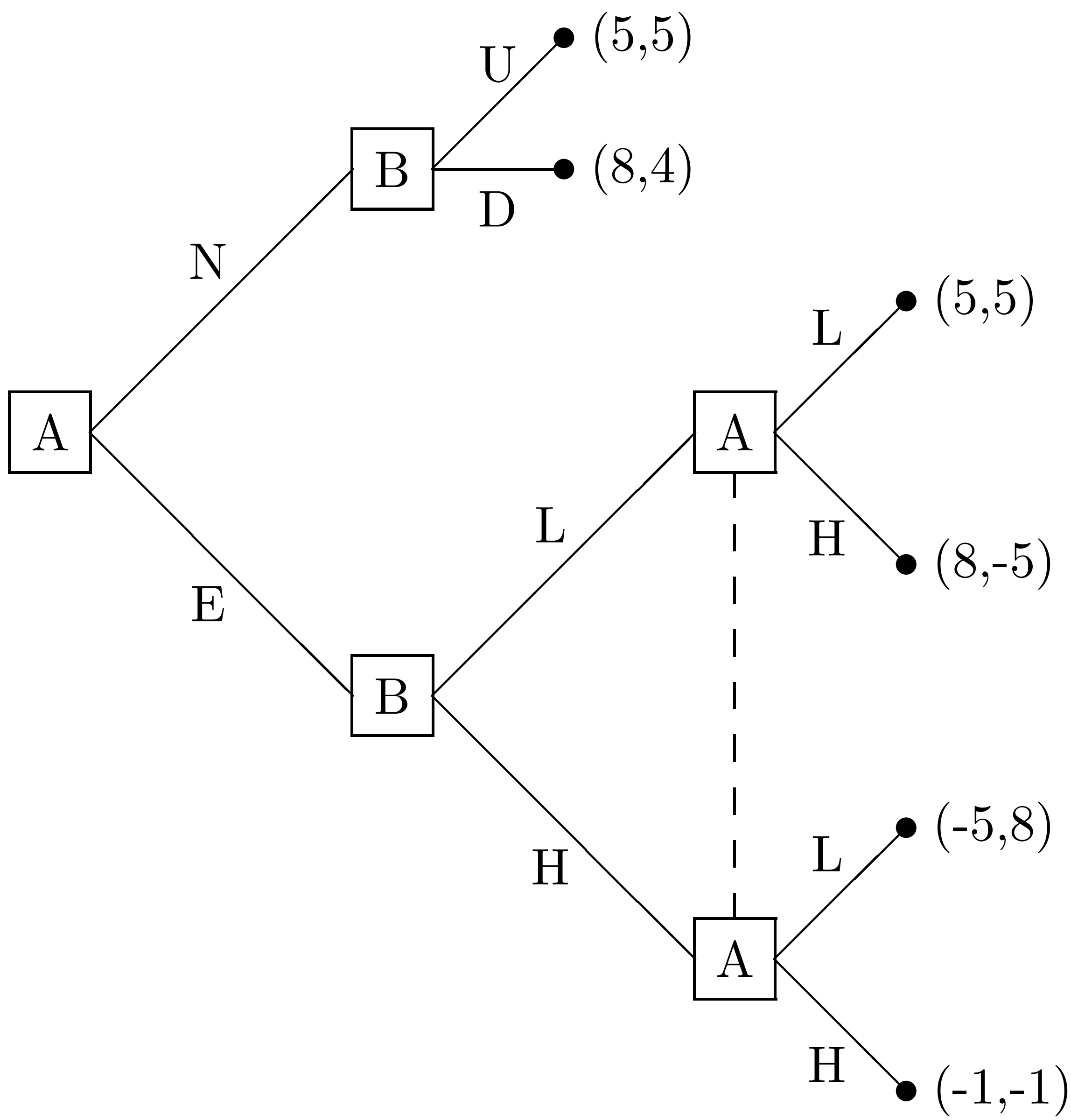
  Here the second player is saying that you better choose U and give me a payoff of 2 or otherwise (if you choose D) then I'll choose L and we'll both get -1. In other words Player 2 is threatening!

  But this threat is not credible. If Player 1 indeed plays D, then Player 2 will play R.
- This observation motivates the concept of **subgame perfect Nash equilibrium**.

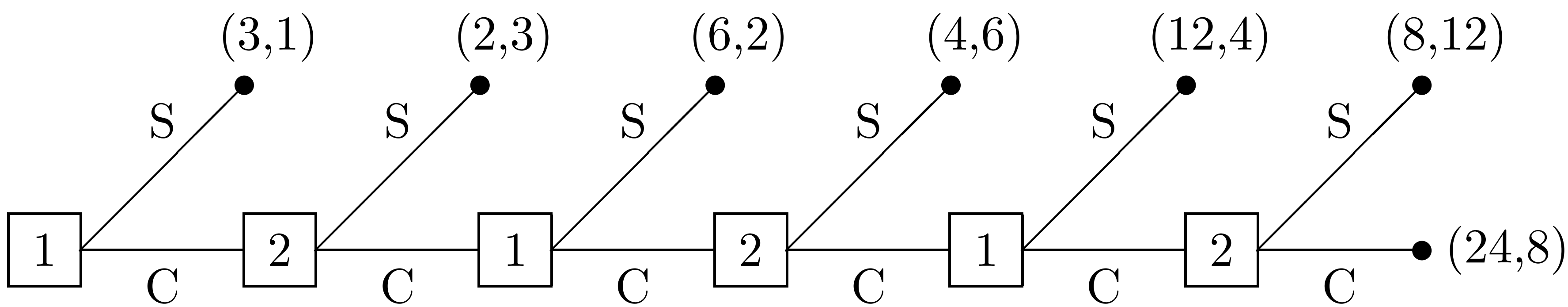
Subgame-Perfect Nash Equilibrium

- Subgame perfect Nash equilibrium can be seen as an extension of the backwards induction method to deal with extensive form games.
- A **proper subgame** is a subset of the nodes of the game starting with an initial node and including all its successors that preserves all information sets of the game and over which a new game is defined by the restriction of the original game elements (i.e., actions, payoffs, information sets, etc.).
- A strategy profile is a **subgame perfect Nash equilibrium** if
  - it is a Nash equilibrium, and moreover
  - for every proper subgame, the restriction of those strategies to the subgame is also a Nash equilibrium.

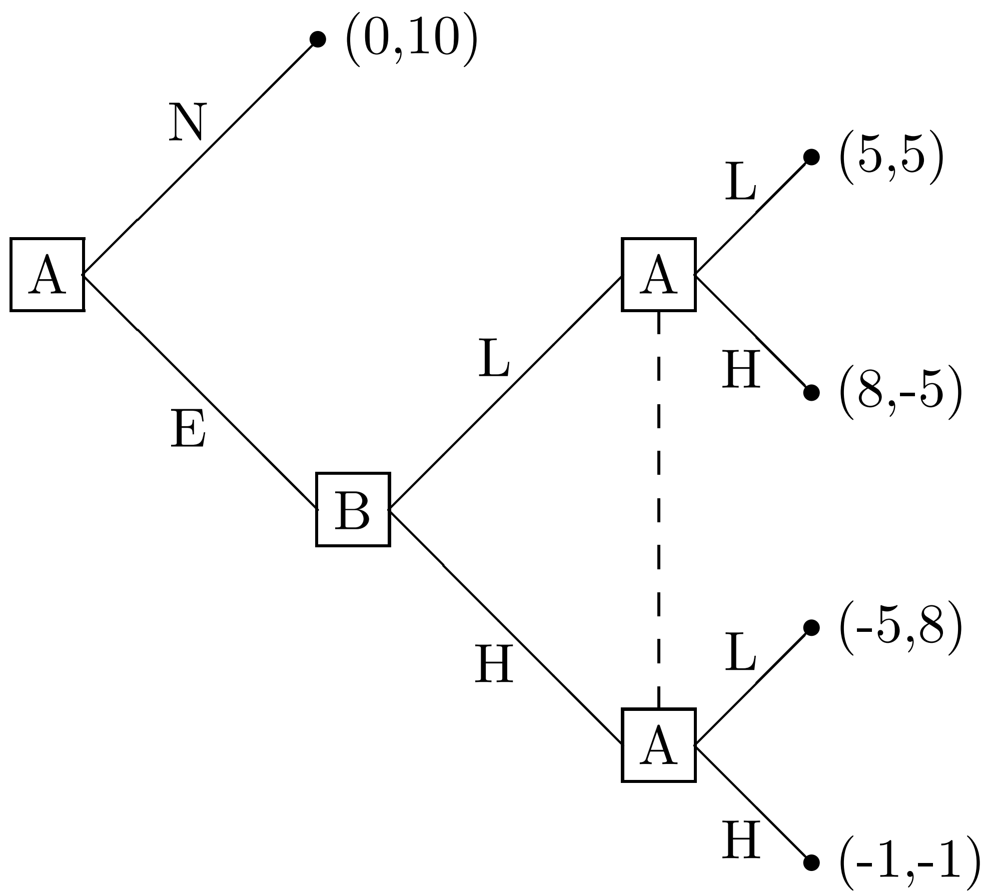
Example:



Example:



Example revisited: Recall that the Nash equilibria of the following game are (NH;H) and (NL;H).



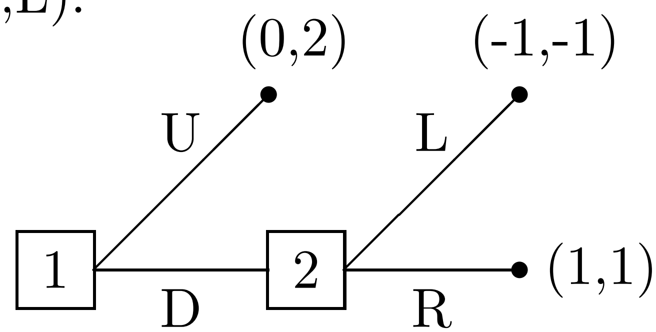
Which of these Nash equilibria are subgame perfect?

- We need to look at the subgame. The matrix form equivalent of the subgame is:

		Firm B	
		L	H
Firm A	L	5,5	-5,8
	H	8,-5	-1,-1

- The unique Nash equilibrium of this subgame is (H;H).
- Since (L;H) is not a Nash equilibrium of the subgame, (NL;H) is not a subgame perfect Nash equilibrium.
- The only subgame perfect Nash equilibrium is (NH;H).

Example revisited: Recall that the two Nash equilibria in the following game are (D;R) and (U;L).

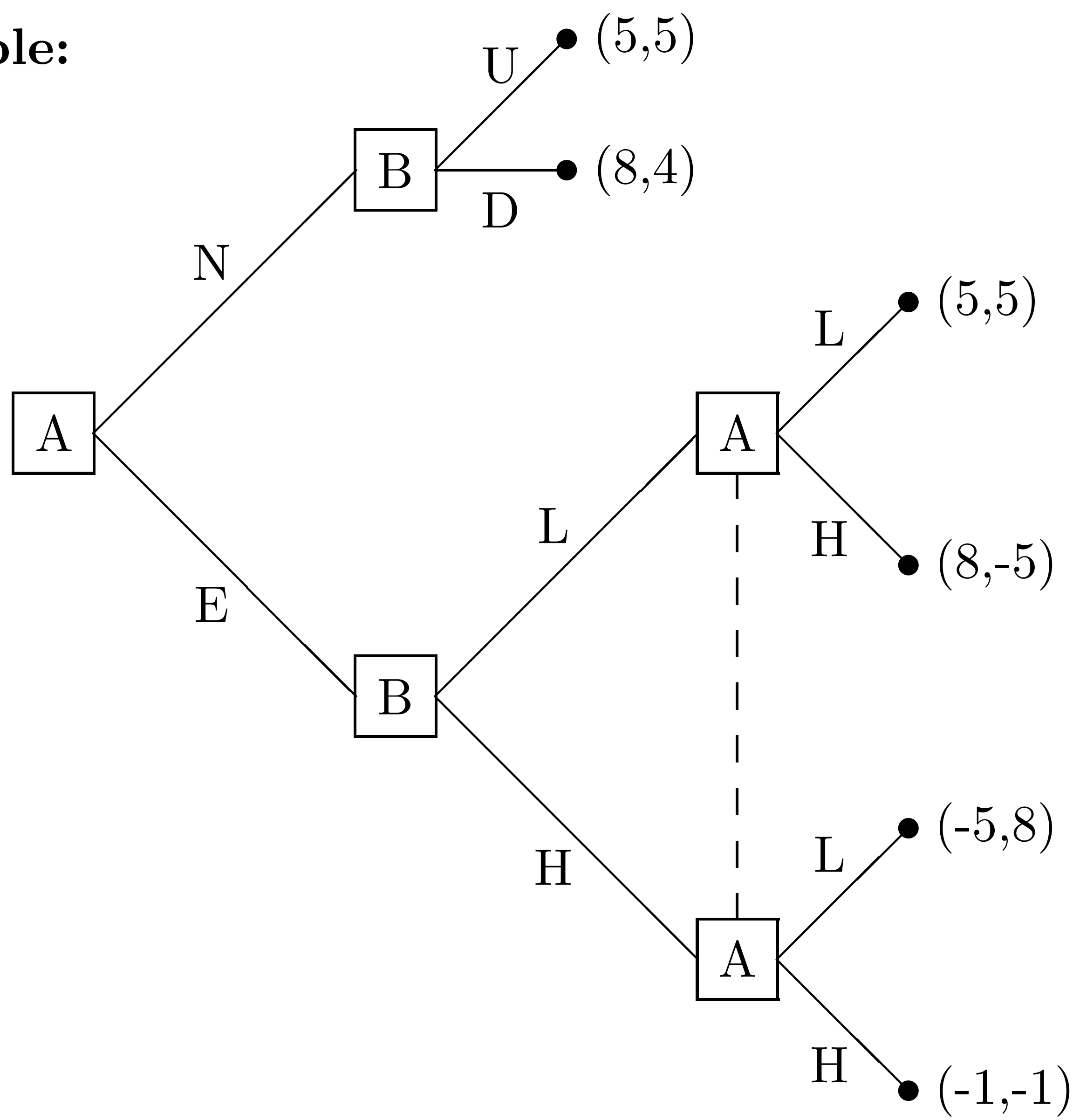


- Consider the subgame (which is simply a one-person decision problem). The unique Nash equilibrium of this one person game is R. Therefore (U;L) is not a subgame perfect Nash equilibrium and hence (D;R) is the only subgame perfect Nash equilibrium.
- **Remark:** Since this is a sequential game we can solve it via backwards induction and this also gives us (D;R). This is not a coincidence. For sequential games subgame perfect Nash equilibria coincide with the outcomes obtained with backwards induction.

Finding Subgame Perfect Nash Eq.

- Finding subgame perfect Nash equilibria is similar to backwards induction:
- We start from the subgames that starts with a node closest to a terminal node, find Nash equilibrium of the subgame, replace the subgame with the Nash equilibrium payoff and work backwards.
- If there are more than one Nash equilibrium of the subgame we repeat this for each subgame.

Example:



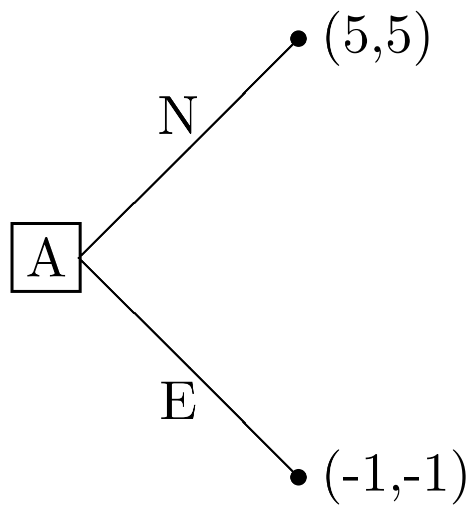
- The first subgame is a 2 person simultaneous game with the following strategic form:

		Firm B	
		L	H
Firm A	L	5,5	-5,8
	H	8,-5	-1,-1

The only Nash equilibrium of this subgame is (H;H) which yields a payoff of (-1,-1).

- The second subgame is a simple 1 person decision problem with the (trivial) Nash equilibrium U. This yields a payoff of (5,5).

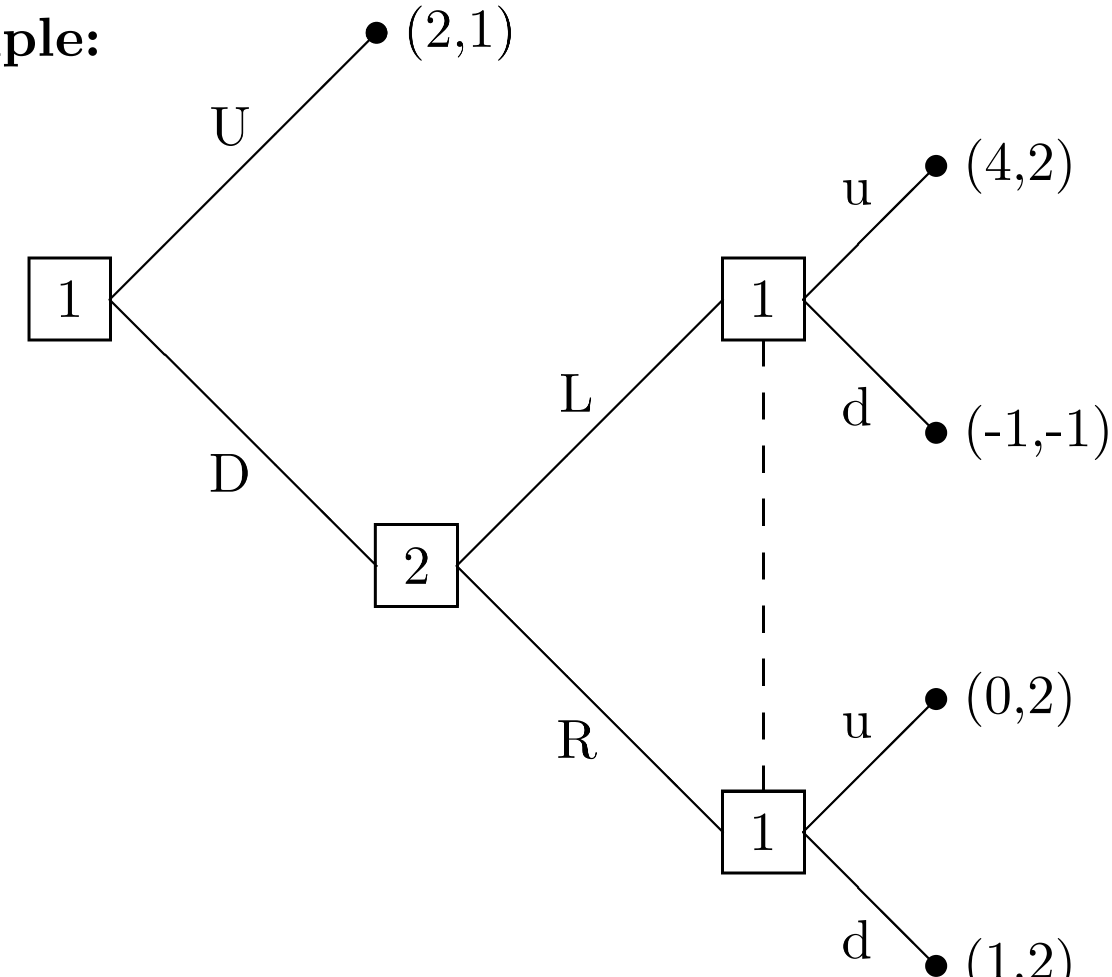
- If we plug these payoffs instead of the subgames we obtain:



- This is a simple 1 person decision problem with the (trivial) Nash equilibrium N. This yields a payoff of (5,5).
- What about the subgame perfect Nash equilibrium strategies?

  All we need to do is keep track of the Nash equilibrium strategies at each step. Therefore the only subgame perfect Nash equilibrium is (NH;UH).

Example:



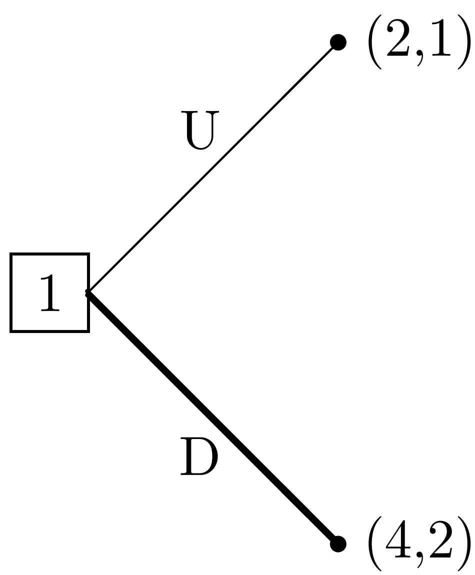
Let's find the subgame perfect Nash equilibria.

We start with the subgame. The strategic equivalent of the subgame is:

		Player 2	
		L	R
Player 1	u	4,2	0,2
	d	-1,-1	1,2

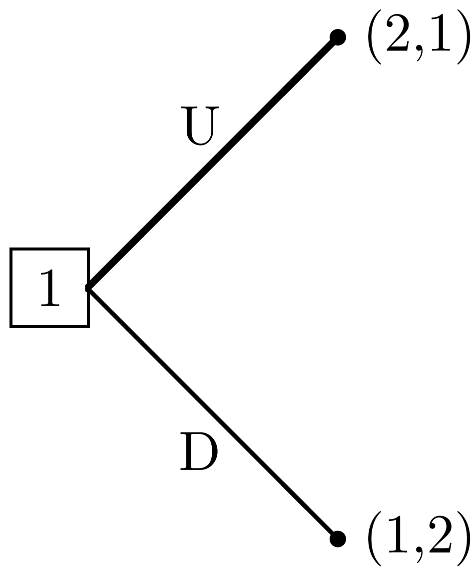
There are two Nash equilibria of this subgame: (u,L) yielding a payoff of (4,2) and (d,R) yielding a payoff of (1,2). We need to consider two cases.

Case 1 (with Nash equilibrium (u,L)): Replacing this Nash equilibrium payoff with the subgame reduces the game to:



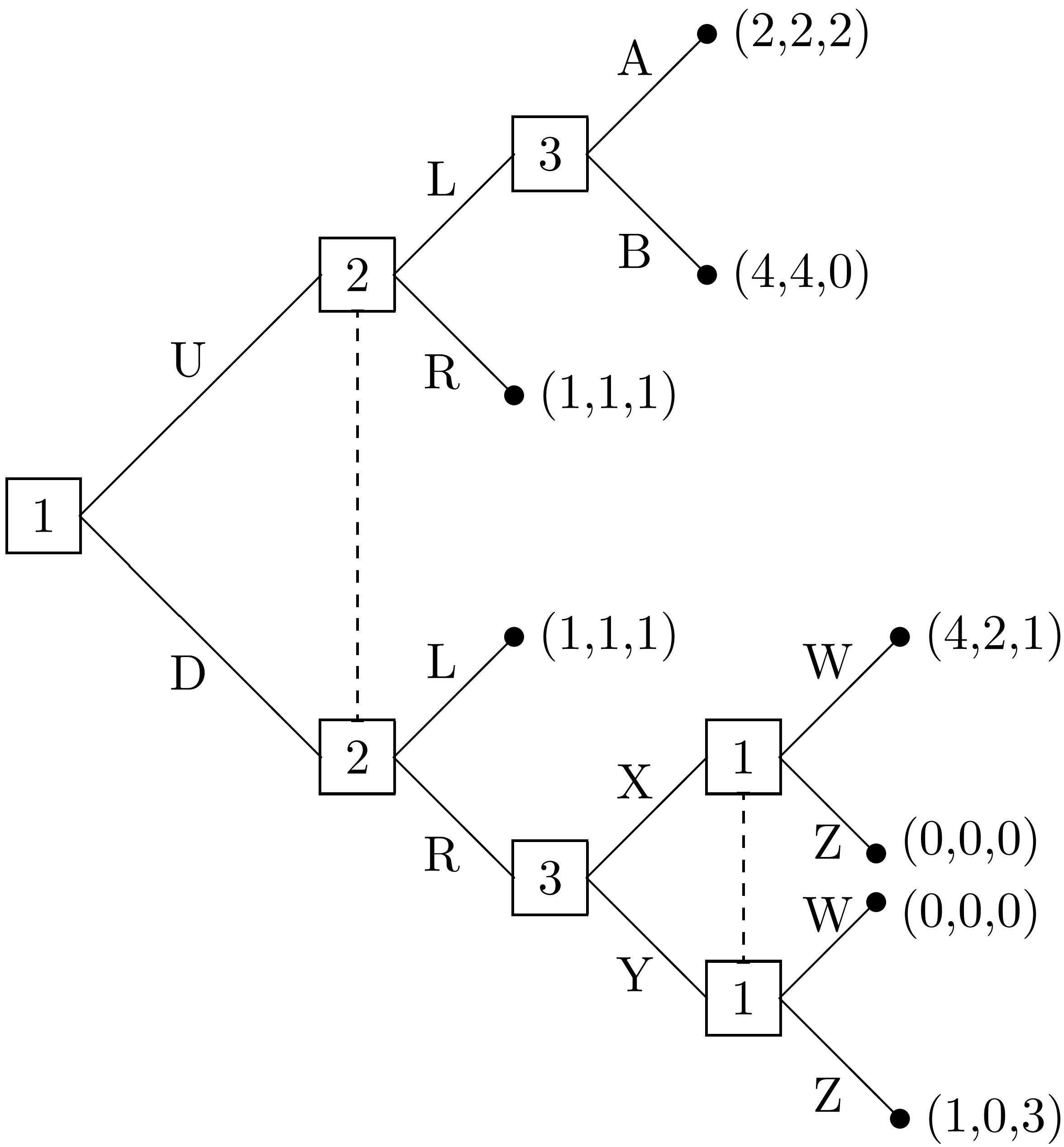
The first subgame perfect Nash equilibrium is (Du,L) yielding (4,2).

Case 2 (with Nash equilibrium (d,R)): Replacing this Nash equilibrium payoff with the subgame reduces the game to:



The second subgame perfect Nash equilibrium is (Ud,R) yielding (2,1).

Example:

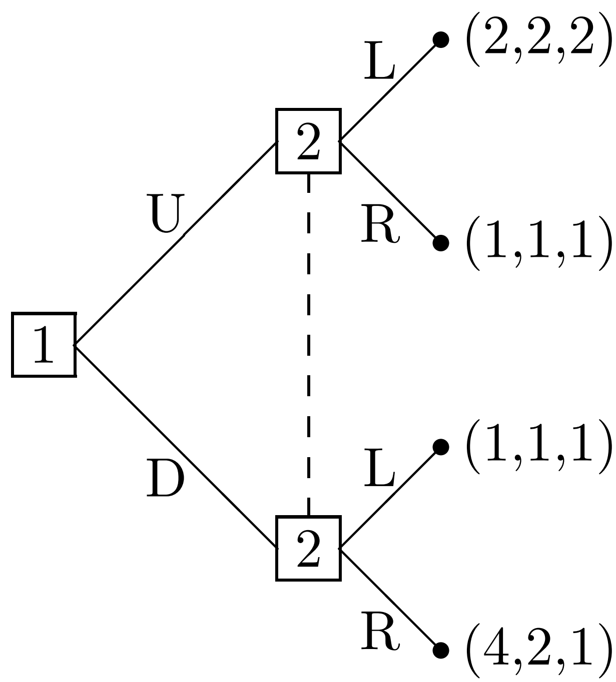


- There are 2 subgames.
- The first subgame is a simple 1 person decision problem with Player 3. The trivial Nash equilibrium for this game is A. This yields a payoff vector of (2,2,2).
- The second subgame is a simultaneous game between Player 1 and Player 3. It has the following strategic form:

		Player 3: X	Player 3: Y
Player 1	W	4,(2),1	0,(0),0
	Z	0,(0),0	1,(0),3

  There are two Nash equilibria of this subgame: (W,X) yielding a payoff of (4,2,1) and (Z,Y) yielding a payoff of (1,0,3).
- Since Subgame 1 has only one equilibrium (A) we can substitute it with the payoff vector (2,2,2).

- Subgame 2 on the other hand has two equilibria. So we should consider both cases.
- **Case 1** (Subgame 2 replaced with the payoff of Nash eq. (W,X)): In this case the game reduces to



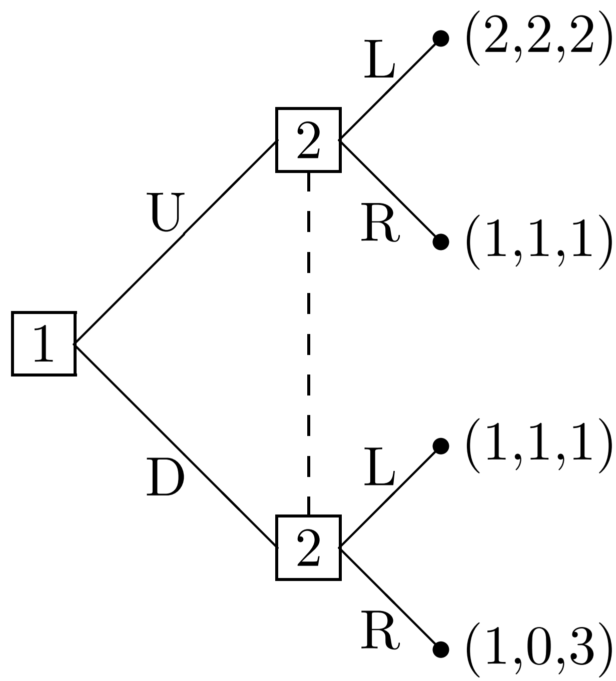
Note that the players do not care for the last payoff. The strategic equivalent of this game is:

		Player 2	
		L	R
Player 1	U	2,2,(2)	1,1,(1)
	D	1,1,(1)	4,2,(1)

There are two Nash equilibria of this game: (U,L) yielding a payoff of (2,2,2) and (D,R) yielding a payoff of (4,2,1).

- Therefore two subgame perfect Nash equilibria results from Case 1:
  - (UW,L,AX) yielding a payoff vector (2,2,2) and
  - (DW,R,AX) yielding a payoff vector (4,2,1).

- **Case 2** (Subgame 2 replaced with the payoff of Nash eq. (Z,Y)): In this case the game reduces to



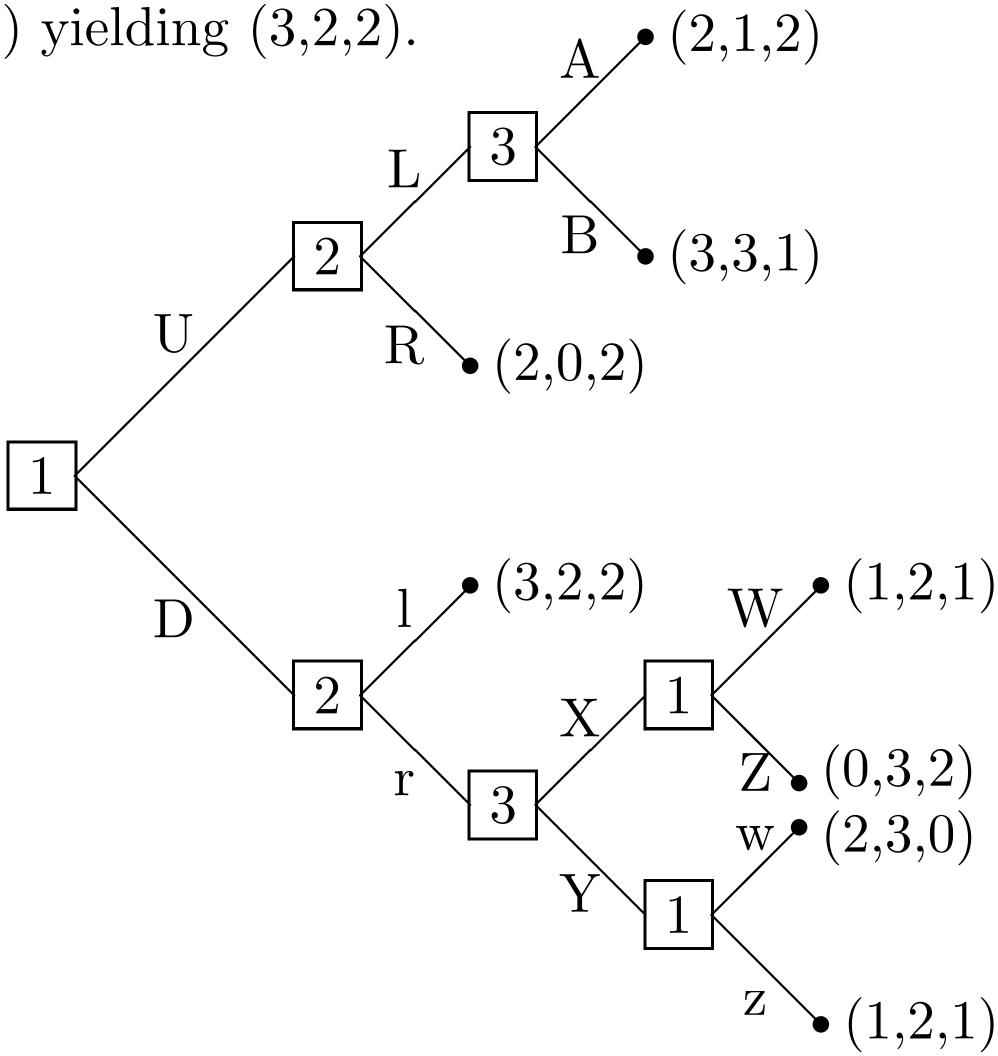
The strategic equivalent of this game is:

		Player 2	
		L	R
Player 1	U	2,2,(2)	1,1,(1)
	D	1,1,(1)	1,0,(3)

There is one Nash equilibrium of this game: (U,L) yielding a payoff of (2,2,2).

- Therefore one subgame perfect Nash equilibria results from Case 2: (UZ,L,AY) yielding a payoff vector (2,2,2).
- Hence there are a total of three subgame perfect Nash equilibria in this game:
  - (UW,L,AX) with (2,2,2),
  - (DW,R,AX) with (4,2,1), and
  - (UZ,L,AY) with (2,2,2).

Example: We used this example to illustrate that backwards induction may cause "problems" if there are ties in payoffs. We can easily use the notion of subgame perfect Nash eqm. here. This game has two s-p-n-equilibrium: (UWw,Lr,AX) yielding (2,1,2) and (DWw,Ll,AX) yielding (3,2,2).



Example: Firm 1 and Firm 2 are the only competitors in a market for a good. The price in the market is given by the inverse demand equation $P = 10 - (Q_1 + Q_2)$ where Q_1 is the output of Firm 1 and Q_2 is the output of Firm 2. Firm 1's total cost function is $C_1 = 4Q_1$ and Firm 2's total cost function is $C_2 = 2Q_2$. Each firm wants to maximize it's profits. First Firm 2 (the leader) chooses Q_2 and next, observing Firm 2's choice, Firm 1 (the follower) chooses Q_1. What will be the subgame perfect Nash equilibrium (the Stackelberg equilibrium) in this market?

We can find the subgame perfect Nash equilibria with backwards induction in this question.

Let Firm 2's choice be Q_2. Firm 1 wants to maximize it's profits

$$\begin{aligned}\Pi_1 = PQ_1 - C_1 &= [10 - (Q_1 + Q_2)]Q_1 - 4Q_1 \\ &= 10Q_1 - Q_1^2 - Q_1 Q_2 - 4Q_1 \\ &= 6Q_1 - Q_1^2 - Q_1 Q_2\end{aligned}$$

Taking the derivative of Π_1 and equating to zero gives

$$6 - 2Q_1 - Q_2 = 0$$

and therefore

$$Q_1 = \frac{6 - Q_2}{2}.$$

This is the equilibrium strategy for Firm 1. Since Firm 1 observes the decision of Firm 2, it can condition its strategy on Firm 2's choice.

Next we should find Q_2. Here note that Firm 2 knows how Firm 1 will respond to its production decision. Firm 2 wants to maximize it's profits:

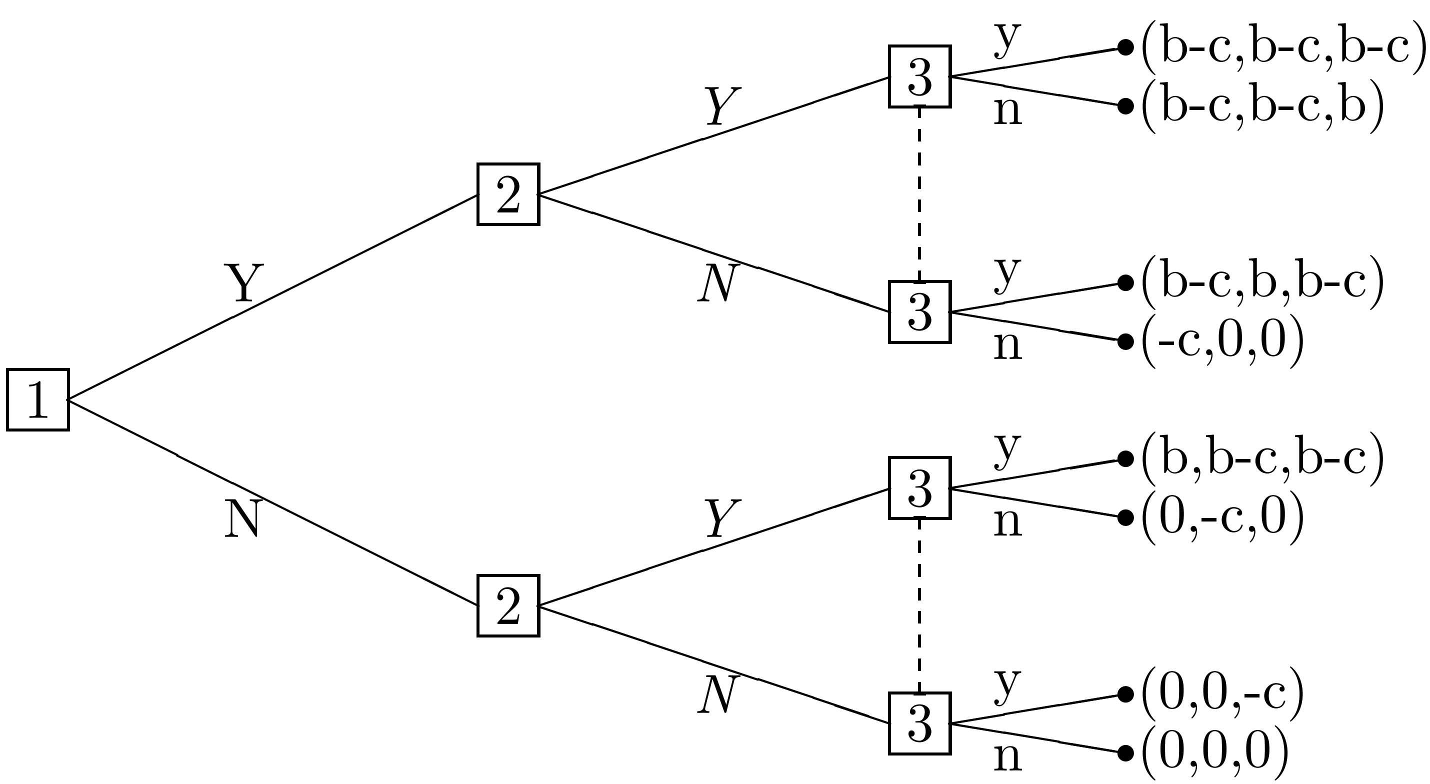
$$\begin{aligned}
\Pi_2 &= PQ_2 - C_2 \\
&= [10 - (Q_1 + Q_2)]Q_2 - 2Q_2 \\
&= [10 - (\frac{6 - Q_2}{2} + Q_2)]Q_2 - 2Q_2 \\
&= 5Q_2 - \frac{1}{2}Q_2^2
\end{aligned}$$

Taking the derivative of Π_2 and equating to zero gives

$$Q_2 = 5$$

Therefore the subgame perfect Nash equilibrium strategies are $(\frac{6-Q_2}{2}, 5)$ and the eventual production is $Q_1 = 0.5$ and $Q_2 = 5$.

Example (Voting for a Pay Raise revisited): Three legislators are voting on whether to give themselves a pay raise. All three want the pay raise; however each face a small cost in voter resentment $c > 0$. The benefit for the raise is $b > c$. Find the subgame perfect Nash equilibrium if legislator 1 votes first and next observing her legislators 2 and 3 vote simultaneously.



Subgame 1:

		Player 3	
		y	n
Player 2	Y	(b-c),b-c,b-c	(b-c),b-c,b
	N	(b-c),b,b-c	(-c),0,0

Two Nash equilibria: (Y,n) and (N,y).

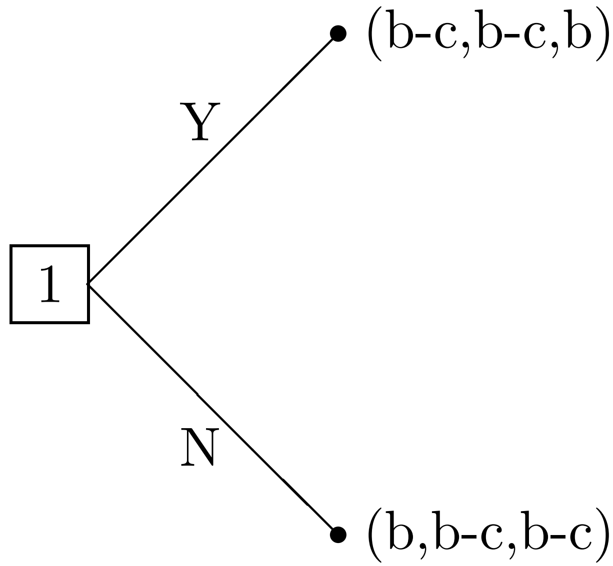
Subgame 2:

		Player 3	
		y	n
Player 2	Y	(b),b-c,b-c	(0),-c,0
	N	(0),0,-c	(0),0,0

Two Nash equilibria: (Y,y) and (N,n).

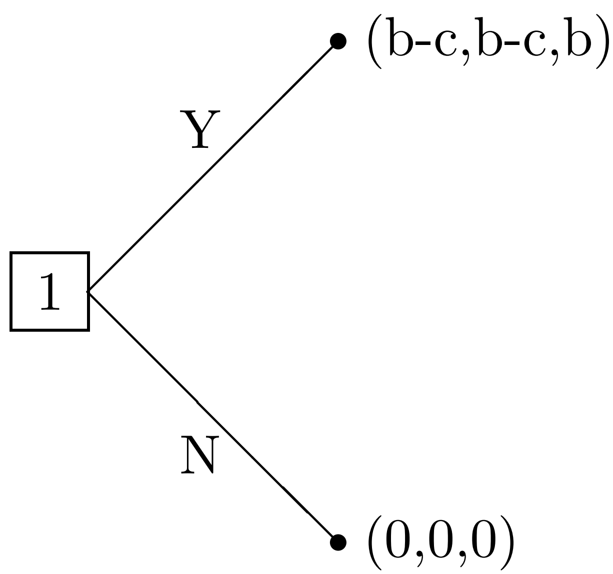
We have 4 cases to consider:

Case 1: Replace Subgame 1 with Nash eqm (Y,n) which yields (b-c,b-c,b) and Subgame 2 with Nash eqm (Y,y) which yields (b,b-c,b-c):



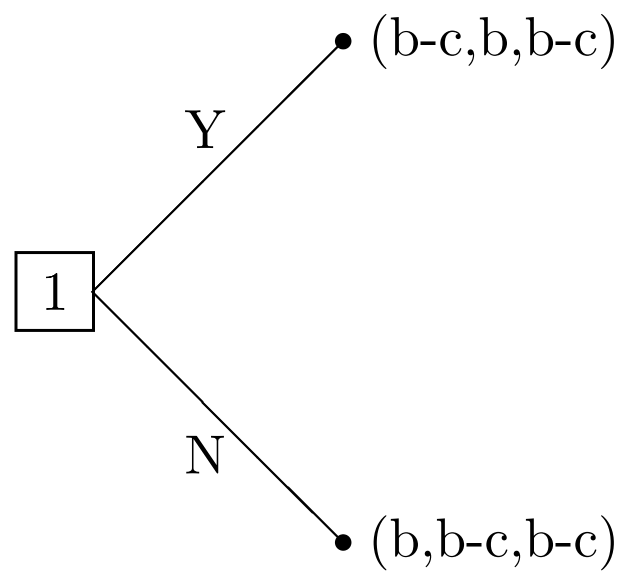
There is only one trivial Nash equilibrium N. Therefore only s-p-n-e for Case 1 is (N, YY,ny) with a payoff of (b,b-c,b-c).

Case 2: Replace Subgame 1 with Nash eqm (Y,n) which yields (b-c,b-c,b) and Subgame 2 with Nash eqm (N,n) which yields (0,0,0):



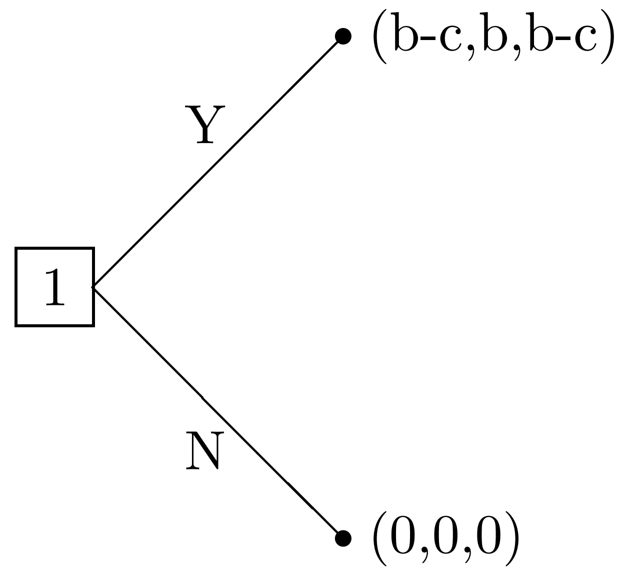
There is only one trivial Nash equilibrium Y. Therefore only s-p-n-e for Case 2 is (Y, YN,nn) with a payoff of (b-c,b-c,b).

Case 3: Replace Subgame 1 with Nash eqm (N,y) which yields (b-c,b,b-c) and Subgame 2 with Nash eqm (Y,y) which yields (b,b-c,b-c):



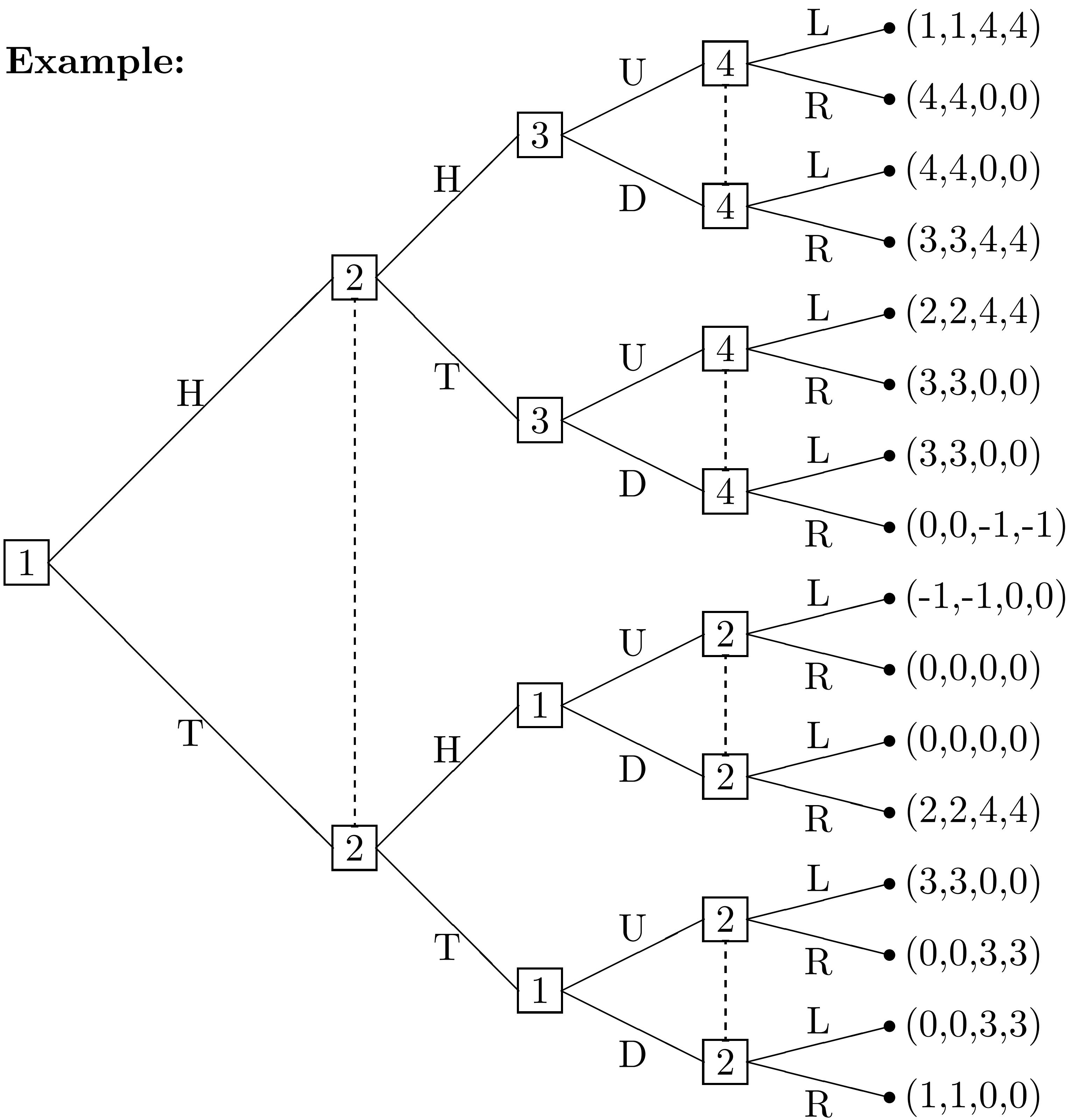
There is only one trivial Nash equilibrium N. Therefore only s-p-n-e for Case 3 is (N,NY,yy) with a payoff of (b,b-c,b-c).

Case 4: Replace Subgame 1 with Nash eqm (N,y) which yields (b-c,b,b-c) and Subgame 2 with Nash eqm (N,n) which yields (0,0,0):



There is only one trivial Nash equilibrium Y. Therefore only s-p-n-e for Case 4 is (Y,NN,yn) with a payoff of (b-c,b,b-c).

Example: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(3,3,0,0)
(0,0,-1,-1)
(-1,-1,0,0)
(0,0,0,0)
(0,0,0,0)
(2,2,4,4)
(3,3,0,0)
(0,0,3,3)
(0,0,3,3)
(1,1,0,0)

Subgame 1:

Player 4

Player 3		L	R
	U	(1),(1),4,4	(4),(4),0,0
	D	(4),(4),0,0	(3),(3),4,4

Two Nash equilibria: (U,L) and (D,R).

Subgame 2:

Player 4

Player 3		L	R
	U	(2),(2),4,4	(3),(3),0,0
	D	(3),(3),0,0	(0),(0),-1,-1

One Nash equilibrium: (U,L).

Subgame 3:

		Player 2	
		L	R
Player 1	U	-1,-1,(0),(0)	0,0,(0),(0)
	D	0,0,(0),(0)	2,2,(4),(4)

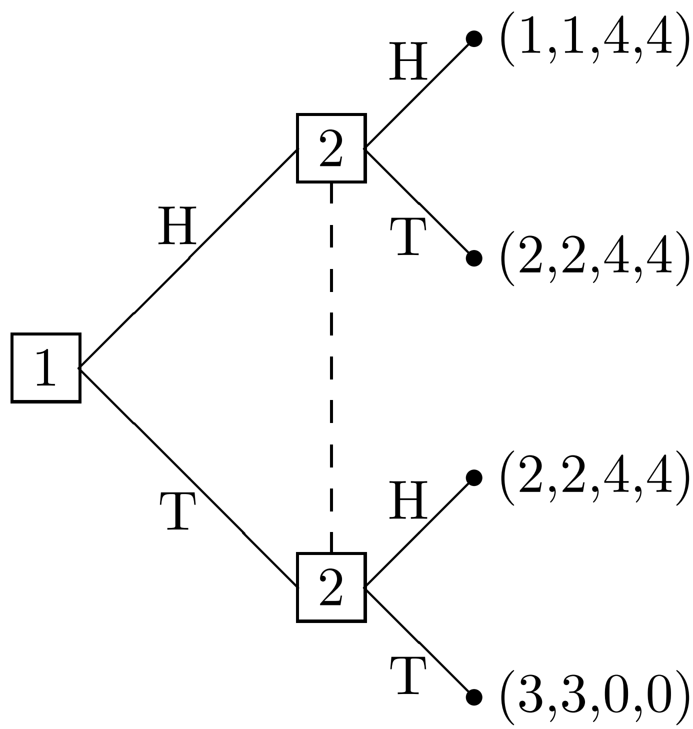
One Nash equilibrium: (D,R).

Subgame 4:

		Player 2	
		L	R
Player 1	U	3,3,(0),(0)	0,0,(3),(3)
	D	0,0,(3),(3)	1,1,(0),(0)

Two Nash equilibria: (U,L) and (D,R).

Case 1: Subgame 1 replaced with (U,L) that yields (1,1,4,4) and Subgame 4 replaced with (U,L) yielding (3,3,0,0):



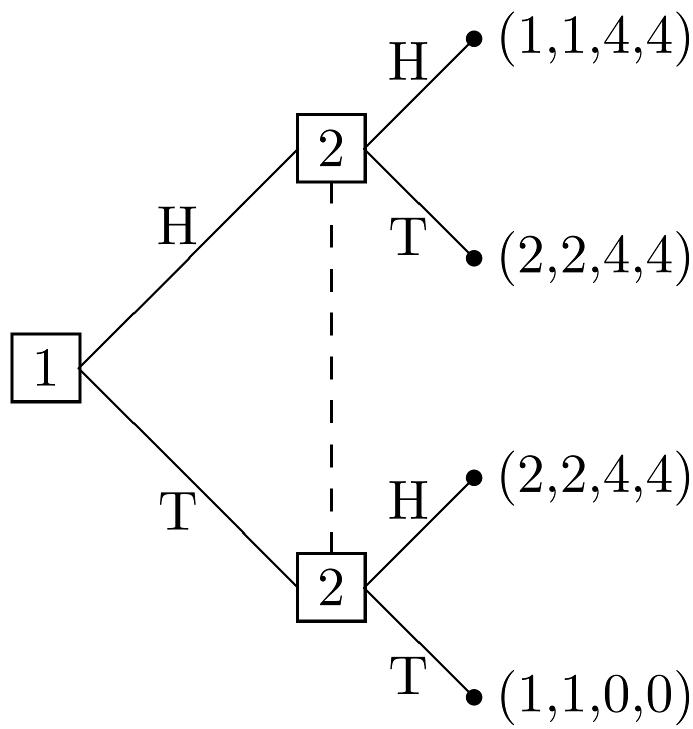
The strategic equivalent of this game is:

		Player 2 H	Player 2 T
Player 1	H	1,1,(4),(4)	2,2,(4),(4)
	T	2,2,(4),(4)	3,3,(0),(0)

This game has 1 Nash equilibrium (T,T). Therefore there is one SPNE for Case 1:

- (TDU,TRL,UU,LL) yielding (3,3,0,0).

Case 2: Subgame 1 replaced with (U,L) that yields (1,1,4,4) and Subgame 4 replaced with (D,R) yielding (1,1,0,0):



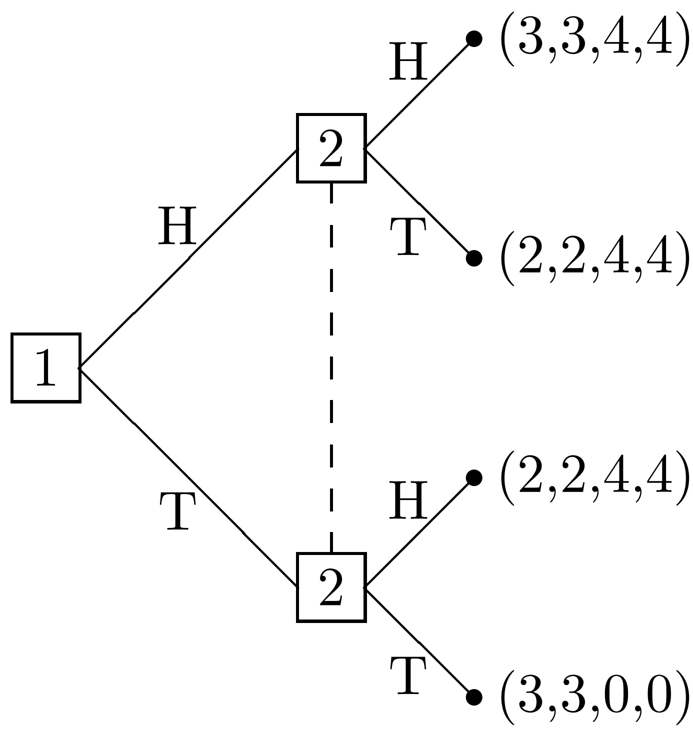
The strategic equivalent of this game is:

		Player 2	
		H	T
Player 1	H	1,1,(4),(4)	2,2,(4),(4)
	T	2,2,(4),(4)	1,1,(0),(0)

This game has 2 Nash equilibria (H,T) and (T,H). Therefore there are 2 SPNE for Case 2:

- (HDD,TRR,UU,LL) yielding (2,2,4,4),
- (TDD,HRR,UU,LL) yielding (2,2,4,4).

Case 3: Subgame 1 replaced with (D,R) that yields (3,3,4,4) and Subgame 4 replaced with (U,L) yielding (3,3,0,0):



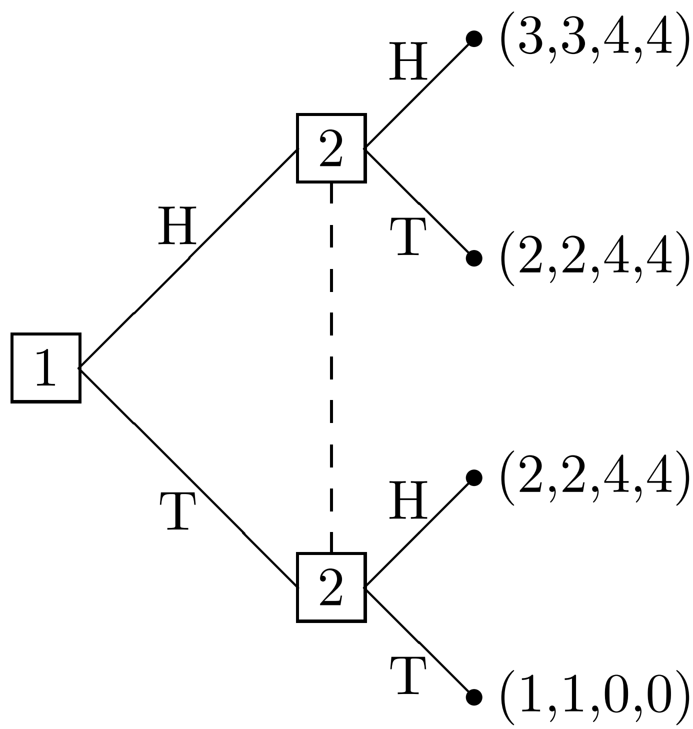
The strategic equivalent of this game is:

		Player 2	
		H	T
Player 1	H	3,3,(4),(4)	2,2,(4),(4)
	T	2,2,(4),(4)	3,3,(0),(0)

This game has 2 Nash equilibria (H,H) and (T,T). Therefore there are 2 SPNE for Case 3:

- (HDU,HRL,DU,RL) yielding (3,3,4,4),
- (TDU,TRL,DU,RL) yielding (3,3,0,0).

Case 4: Subgame 1 replaced with (D,R) that yields (3,3,4,4) and Subgame 4 replaced with (D,R) yielding (1,1,0,0):



The strategic equivalent of this game is:

		Player 2	
		H	T
Player 1	H	3,3,(4),(4)	2,2,(4),(4)
	T	2,2,(4),(4)	1,1,(0),(0)

This game has 1 Nash equilibrium (H,H). Therefore there is one SPNE for Case 4:

- (HDD,HRR,DU,RL) yielding (3,3,4,4).