How to Solve Strategic Games?

There are three main concepts to solve strategic games:

1. Dominant Strategies & Dominant Strategy Equilibrium
2. Dominated Strategies & Iterative Elimination of Dominated Strategies
3. Nash Equilibrium

Dominant Strategies

- A strategy is a **dominant strategy** for a player if it yields the best payoff (for that player) no matter what strategies the other players choose.
- If all players have a dominant strategy, then it is natural for them to choose the dominant strategies and we reach a **dominant strategy equilibrium**.

Example (Prisoner's Dilemma):

		Prisoner 2	
		Confess	Deny
Prisoner 1	Confess	-10, -10	-1, -25
	Deny	-25, -1	-3, -3

Confess is a dominant strategy for both players and therefore (Confess,Confess) is a dominant strategy equilibrium yielding the payoff vector (-10,-10).

Example (Time vs. Newsweek):

		Newsweek	
		AIDS	BUDGET
Time	AIDS	35,35	70,30
	BUDGET	30,70	15,15

The AIDS story is a dominant strategy for both Time and Newsweek. Therefore (AIDS,AIDS) is a dominant strategy equilibrium yielding both magazines a market share of 35 percent.

Example:

		Player 2: X	Player 2: Y
Player 1	A	5,2	4,2
	B	3,1	3,2
	C	2,1	4,1
	D	4,3	5,4

- Here Player 1 does not have a single strategy that "beats" every other strategy. Therefore she does not have a dominant strategy.
- On the other hand Y is a dominant strategy for Player 2.

Example (with 3 players):

P3: A

	P2: L	P2: R
P1: U	3,2,1	2,1,1
P1: M	2,2,0	1,2,1
P1: D	3,1,2	1,0,2

P3: B

	P2: L	P2: R
P1: U	1,1,2	2,0,1
P1: M	1,2,0	1,0,2
P1: D	0,2,3	1,2,2

Here

- U is a dominant strategy for Player 1, L is a dominant strategy for Player 2, B is a dominant strategy for Player 3,
- and therefore (U;L;B) is a dominant strategy equilibrium yielding a payoff of (1,1,2).

Dominated Strategies

- A strategy is **dominated** for a player if she has another strategy that performs at least as good no matter what other players choose.
- Of course if a player has a dominant strategy then this player's all other strategies are dominated. But there may be cases where a player does not have a dominant strategy and yet has dominated strategies.

Example:

		Player 2 X	Player 2 Y
Player 1	A	5,2	4,2
	B	3,1	3,2
	C	2,1	4,1
	D	4,3	5,4

- Here B & C are dominated strategies for Player 1 and
- X is a dominated strategy for Player 2.

Therefore it is natural for

- Player 1 to assume that Player 2 will not choose X, and
- Player 2 to assume that Player 1 will not choose B or C.

Therefore the game reduces to

		Player 2
		Y
Player 1	A	4,2
	D	5,4

In this reduced game D dominates A for Player 1. Therefore we expect players choose (D;Y) yielding a payoff of (5,4).

This procedure is called **iterated elimination of dominated strategies**.

Example:

Player 2

		L	R
	U	10,5	10,10
Player 1	M	20,10	30,5
	D	30,10	5,5

U is dominated for Player 1 $\Longrightarrow$ Eliminate.

		Player 2	
		L	R
Player 1	M	20,10	30,5
	D	30,10	5,5

R is dominated for Player 2 $\Longrightarrow$ Eliminate.

		Player 2
		L
Player 1	M	20,10
	D	30,10

M is dominated for Player 1 $\Longrightarrow$ Eliminate and (D;L) survives.

Example:

		Player 2				
		V	W	X	Y	Z
	A	4,-1	3,0	-3,1	-1,4	-2,0
	B	-1,1	2,2	2,3	-1,0	2,5
Player 1	C	2,1	-1,-1	0,4	4,-1	0,2
	D	1,6	-3,0	-1,4	1,1	-1,4
	E	0,0	1,4	-3,1	-2,3	-1,-1

Here the order of elimination is: D-V-E-W-A-Y-C-X and hence (B;Z) survives the elimination yielding a payoff of (2,5).

Example: Each of two players announces an integer between 0 and 100. Let a_1 be the announcement of Player 1 and a_2 be the announcement of Player 2. The payoffs are determined as follows:

- If $a_1 + a_2 \leq 100$: Player 1 receives a_1 and Player 2 receives a_2;
- If $a_1 + a_2 > 100$ and $a_1 > a_2$: Player 1 receives $100 - a_2$ and Player 2 receives a_2;
- If $a_1 + a_2 > 100$ and $a_1 < a_2$: Player 1 receives a_1 and Player 2 receives $100 - a_1$;
- If $a_1 + a_2 > 100$ and $a_1 = a_2$: Both players receive 50.

Solve this game with iterated elimination of dominated strategies.

Observation: If Player 1 announces 51 her payoff is

- 50 if Player 2 announces 50 or 51, and
- 51 if Player 2 announces anything else.

Likewise for Player 2.

Round 1: If Player 1 announces $a_1 < 51$ she'll get a_1 no matter what Player 2 announces. Therefore any strategy smaller than 51 is dominated by 51. Likewise for Player 2. We can delete all strategies between 0 and 50 for both players.

Round 2: If Player 1 announces 100 she can get at most 50. This is because Player 2 announces a number between 51-100. Therefore 100 is dominated by 51 in this reduced game. Likewise for Player 2. We can delete 100 for both players.

Round 3: If Player 1 announces 99 she can get at most 50. This is because Player 2 announces a number between 51-99. Therefore 99 is dominated by 51 in this further reduced game. Likewise for Player 2. We can delete 99 for both players.

$$\vdots \qquad\qquad \vdots$$

Round 49: If Player 1 announces 53 she can get at most 50. This is because Player 2 announces a number between 51-53. Therefore 53 is dominated by 51 in this further, further, . . . , further reduced game. Likewise for Player 2. We can delete 53 for both players.

Round 50: If Player 1 announces 52 she can get at most 50. This is because Player 2 announces a number between 51-52. Therefore 52 is dominated by 51 in this further, further, . . . , further reduced game. Likewise for Player 2. We can delete 52 for both players.

Hence only 51 survives the iterated elimination of strategies for both players. As a result the payoff of each player is 50.

Nash Equilibrium

- In many games there will be no dominant and/or dominated strategies. Even if there is, iterative elimination of dominated strategies will usually not result in a single strategy profile.
- Consider a strategic game. A strategy profile is a **Nash equilibrium** if no player wants to unilaterally deviate to another strategy, given other players' strategies.

Example:

		Player 2 L	Player 2 R
Player 1	U	5, 5	2, 1
	D	4, 7	3, 6

Consider the strategy pair (U;L).

- If Player 1 deviates to D then his payoff reduces to 4.
- If Player 2 deviates to R then her payoff reduces to 1.
- Hence neither player can benefit by a unilateral deviation.
- Therefore (U;L) is a Nash equilibrium yielding the payoff vector (5,5).

Example: Consider the following 3-person simultaneous game. Here Player 1 chooses between the rows U and D, Player 2 chooses between the columns L and R, and Player 3 chooses between the matrices A and B.

P3

A (P2)

P1	L	R
U	5,5,1	2,1,3
D	4,7,6	1,8,5

B (P2)

P1	L	R
U	0,2,2	4,4,4
D	1,1,1	3,7,1

- In this game (U;R;B) is the only Nash equilibrium.

Example (Battle of the Sexes): The following game has two Nash equilibria (U;L) and (D;R).

		Player 2	
		L	R
Player 1	U	3, 1	0, 0
	D	0, 0	1, 3

Example (Matching Pennies): The following game has no Nash equilibrium.

		Player 2: L	Player 2: R
Player 1	U	1, -1	-1, 1
	D	-1, 1	1, -1

Tricks for Finding Nash Equilibrium in Complicated Games

Example:

P2

P1		V	W	X	Y	Z
	A	4,-1	4,2	-3,1	-1,2	-2,0
	B	-1,1	2,2	2,3	-1,0	2,5
P1	C	2,3	-1,-1	0,4	4,-1	0,2
	D	1,3	4,4	-1,4	1,1	-1,2
	E	0,0	1,4	-3,1	-2,3	-1,-1

- In column V, if there is a Nash eqm at all it should be (A;V); otherwise P1 deviates. But it is not a Nash eqm since P2 deviates.
- In column W, if there is a Nash eqm at all it should be (A;W) or (D;W); otherwise P1 deviates. Since P2 does not deviate in either both strategy profiles are Nash eqm.
- In column X, if there is a Nash eqm at all it should be (B;X); otherwise P1 deviates. But it is not a Nash eqm since P2 deviates.
- In column Y, if there is a Nash eqm at all it should be (C;Y); otherwise P1 deviates. But it is not a Nash eqm since P2 deviates.
- In column Z, if there is a Nash eqm at all it should either be (B;Z); otherwise P1 deviates. Since P2 does not deviate here it is a Nash eqm.

Best Response Function

- The following restatement of Nash equilibrium is sometimes useful.

- Consider an n-person strategic game. Let $u_i(s_1^*, \ldots, s_n^*)$ denote the payoff of Player i for the strategy-tuple $(s_1^*, \ldots, s_n^*)$.

  A strategy s_i^* is a **best response** for Player i to the strategy $(s_1^*, \ldots, s_{i-1}^*, s_{i+1}^*, \ldots, s_n^*)$ (of other players) if

$$u_i(s_1^*, \ldots, s_{i-1}^*, s_i^*, s_{i+1}^*, \ldots, s_n^*) \geq u_i(s_1^*, \ldots, s_{i-1}^*, \tilde{s}_i, s_{i+1}^*, \ldots, s_n^*) \quad \text{for any } \tilde{s}_i.$$

  In other words, a strategy s^* is a best response for Player i for the strategy choice $(s_1^*, \ldots, s_{i-1}^*, s_{i+1}^*, \ldots, s_n^*)$ of other players, if it gives the best payoff (ties are allowed) for Player i.

- If we find the best response for Player i for *every* possible strategy choice of other players, the we obtain Player i's **best response function**.

  Note that, for some strategy choices of other players, a player may have more than one best response.

- A strategy profile $(s_1, \ldots, s_n)$ is a *Nash equilibrium*, if the strategy s_i is a best response to strategy

$$(s_1, \ldots, s_{i-1}, s_{i+1}, \ldots, s_n)$$

  for every Player i.

  In other words, every intersection of the best response functions is a Nash equilibrium.

Example: Each of the two players chooses a real number between 0 and 100. Let s_1 denote the choice of Player 1 and s_2 denote the choice of Player 2. The payoffs are determined as follows:

- If $s_1 + s_2 \leq 100$ then Player 1 receives s_1 and Player 2 receives s_2.
- If $s_1 + s_2 > 100$ then both receive 0.

Each player cares for only his/her payoff. Find the Nash equilibria of this game.

We should first find the best response function for both players.

Let $B_1(s_2)$ denote the best response of Player 1 for strategy choice s_2 of Player 2, and $B_2(s_1)$ denote the best response of Player 2 for strategy choice s_1 of Player 1.

$$B_1(s_2) = \begin{cases} 100 - s_2 & \text{if } s_2 \in [0, 100) \\ \text{any strategy} & \text{if } s_2 = 100 \end{cases}$$

$$B_2(s_1) = \begin{cases} 100 - s_1 & \text{if } s_1 \in [0, 100) \\ \text{any strategy} & \text{if } s_1 = 100 \end{cases}$$

Any strategy-pair (s_1, s_2) with $s_1 + s_2 = 100$ is at the intersection of both best response functions and therefore any such pair is a Nash equilibrium.

Example (Oligopoly): Firm 1 and Firm 2 are the only competitors in a market for a good. The price in the market is given by the inverse demand equation $P = 10 - (Q_1 + Q_2)$ where Q_1 is the output of Firm 1 and Q_2 is the output of Firm 2. Firm 1's total cost function is $C_1 = 4Q_1$ and Firm 2's total cost function is $C_2 = 2Q_2$. Each firm wants to maximize it's profits and they simultaneously choose their quantities. What will be the (Cournout) Nash equilibrium in this market?

Firm 1 wants to maximize it's profits

$$\begin{aligned}\Pi_1 = PQ_1 - C_1 &= [10 - (Q_1 + Q_2)]Q_1 - 4Q_1 \\ &= 10Q_1 - Q_1^2 - Q_1Q_2 - 4Q_1 \\ &= 6Q_1 - Q_1^2 - Q_1Q_2\end{aligned}$$

Taking the derivative of Π_1 and equating to zero gives

$$6 - 2Q_1 - Q_2 = 0$$

and therefore

$$Q_1 = \frac{6 - Q_2}{2}.$$

This is Firm 1's best response function. It gives how much Firm 1 should produce depending on Firm 2's production.

Similarly Firm 2 wants to maximize it's profits

$$\begin{aligned}\Pi_2 = PQ_2 - C_2 &= [10 - (Q_1 + Q_2)]Q_2 - 2Q_2 \\ &= 8Q_2 - Q_2^2 - Q_1 Q_2\end{aligned}$$

Taking the derivative of Π_2 and equating to zero gives

$$8 - 2Q_2 - Q_1 = 0$$

and therefore

$$Q_2 = \frac{8 - Q_1}{2}.$$

This is Firm 2's best response function and it gives how much Firm 2 should produce depending on Firm 1's production.

Now that we have two equations in two unknowns, (i.e. Q_1 and Q_2) we can solve them simultaneously:

$$Q_1 = \frac{6 - Q_2}{2} = 3 - \frac{8 - Q_1}{4} = 1 + \frac{Q_1}{4} \Longrightarrow Q_1 = \frac{4}{3}$$

and

$$Q_2 = \frac{8 - \frac{4}{3}}{2} = \frac{10}{3}$$

Since $(Q_1, Q_2) = (4/3, 10/3)$ is on both best response functions, none of the firms wants to deviate to another quantity and hence we have a (Cournout) Nash equilibrium.