Cost Sharing

- Whenever a group undertakes a joint enterprise benefits should be distributed.
- Doctors who share an office share joint costs.

Cost Sharing Between Two Towns

Two nearby towns are considering whether to built a joint water distribution system. These are the options:

1. Town A (population 36,000) could build its own facility for $11 million without any assistance from B.
2. Town B (population 12,000) could build its own facility for $7 million without any assistance from A.
3. A facility that jointly serves both communities would cost $15 million.

If they decide on the third option (which is clearly a good idea!), how should they share the cost?

- *Equal division*? (i.e. \$7.5 million each)

  B is not happy with this proposal!

- *Equal division per capita*?

  Per capita cost $= \$\frac{15{,}000{,}000}{48{,}000} = \$312.5$ Therefore

  - A pays $36{,}000 \times 312.5 = \$11.25$ million
  - B pays $12{,}000 \times 312.5 = \$3.75$ million

  A is not happy with this proposal!

We should focus on how much they *save* instead of how much they pay.

They save a total of \$3 million. How should they share these savings?

- They can *share the savings equally.*

  In this case each save \$1.5 million. Therefore A pays \$9.5 million and B pays \$5.5 million.

- They can share the savings equally among all residents.

  Savings per each resident $= \$\frac{3{,}000{,}000}{48{,}000} = \$62.5$

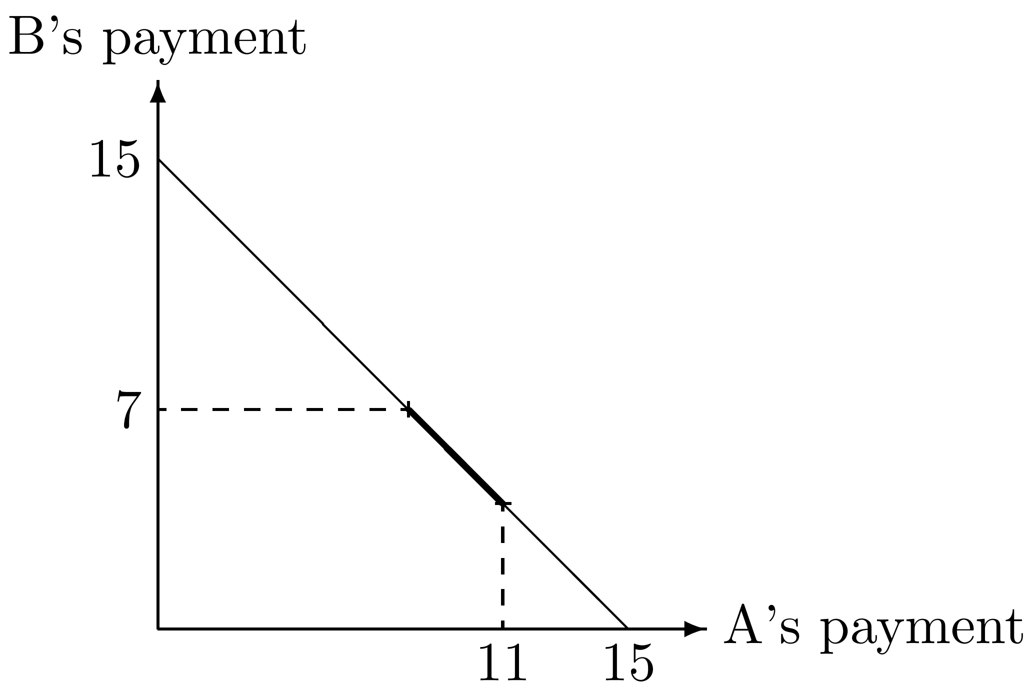
  Therefore A saves $36{,}000 \times 62.5 = \$2.25$ million paying 11-2.25 $= \$8.75$ million and B saves $12{,}000 \times 62.5 = \$0.75$ million paying 7-0.75 = \$ 6.25 million

- They can *share the savings proportional to their costs.* (This is same as sharing the costs proportionally.)

  In this case A pays $\frac{11}{11+7} \times 15 = \$9.17$ million and B pays $\frac{7}{11+7} \times 15 = \$5.83$ million.

  All three approaches give the towns the incentives to cooperate; since they each realize positive savings.

Any allocation highlighted below has the same property:



The set of allocations with this property is called the **core**.

Cost Sharing Between Three Towns

Now let's include a third town C. The new costs (in millions) are as follows:

$$\begin{array}{lll} \text{cost(A)} = 11 & \text{cost(AB)} = 15 & \\ \text{cost(B)} = 7 & \text{cost(AC)} = 14 & \text{cost(ABC)} = 20 \\ \text{cost(C)} = 8 & \text{cost(BC)} = 13 & \end{array}$$

- How should they divide the costs?
- Suppose they divide it proportional to their individual costs. Then:
  - A pays $\frac{11}{11+7+8} \times 20 = \$8.46$ million
  - B pays $\frac{7}{11+7+8} \times 20 = \$5.38$ million
  - C pays $\frac{8}{11+7+8} \times 20 = \$6.15$ million

- But then A & C together pay \$14.61 million whereas they can built their own system for \$14 million!

  The proportional allocation does not respect the opportunity costs of groups of participants.

- The **core** of this cost allocation problem is the set of all divisions of \$20 million such that no town, or group of towns pay more than its opportunity cost.

  That is:

  - A should pay no more than 11
  - B should pay no more than 7
  - C should pay no more than 8
  - A & B should pay no more than 15
  - A & C should pay no more than 14
  - B & C should pay no more than 13

Given that A, B, and C together pay 20, this is equivalent to the following:

- A should pay no less than 7 and no more than 11
- B should pay no less than 6 and no more than 7
- C should pay no less than 5 and no more than 8

A Cooperative Game Model

- Consider a group of parties who want to divide the cost of a common facility. Each party $i = 1, 2, \ldots, n$ has a **stand-alone cost** $c(i)$ it needs to pay if it does not cooperate with the others. Similarly, each subgroup S of parties has a stand-alone cost $c(S)$ they need to pay if they cooperate with each other but not with the remaining parties. This is called a **cost sharing game**.
- A **cost sharing rule** allocates the total cost among the members of a group for every possible cost sharing game.
- An allocation is in the **core** of a cost sharing game, if no participant or group of participants pays more than its stand alone cost.
- As the earlier example shows the core may be quite small. In fact, it may even be empty:

Example:

$$\begin{array}{lll} \text{cost(A)} = 11 & \text{cost(AB)} = 15 & \\ \text{cost(B)} = 7 & \text{cost(AC)} = 13 & \text{cost(ABC)} = 20 \\ \text{cost(C)} = 8 & \text{cost(BC)} = 10 & \end{array}$$

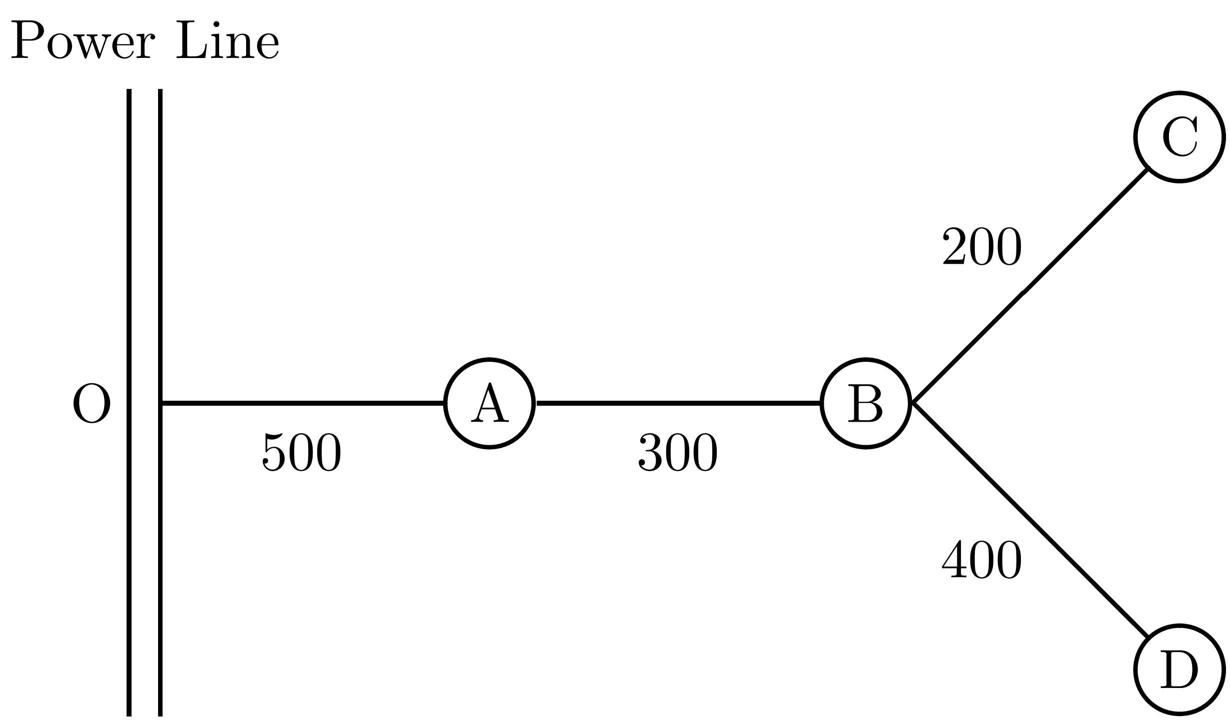
For an allocation to be in the core:

1. A & C together should not pay more than 13 and therefore B should pay at least 7,
2. B & C together should not pay more than 10 and therefore A should pay at least 10,
3. 1 & 2 together imply that A & B together should pay at least 17,
4. which contradicts the requirement that A & B together should not pay more than 15!

In other words, all core inequalities cannot be satisfied in this case.

The Decomposition Principle

Example: The following picture shows the cost of connecting houses A, B, C, and D to the power line. How should we divide the total cost between the houses?



A reasonable way to proceed is to decompose the cost of the project into its constituent parts.

- Everyone uses the line OA, hence divide it equally;
- everyone but A uses AB, hence divide it equally between B,C,D;
- only C uses BC so C should pay for it; and
- only D uses BD so D should pay for it.

The following table summarizes this idea:

	OA	AB	BC	BD	Total
A	125	0	0	0	125
B	125	100	0	0	225
C	125	100	200	0	425
D	125	100	0	400	625

- **Decomposition Principle:** If a cost sharing game decomposes into distinct cost elements, divide the cost of each element equally among those who use it. The charge to each user is its share of each cost element, summed over all elements.
- **Theorem:** If a cost sharing game decomposes into distinct elements, the decomposition principle yields an allocation in the core.

The Shapley Value

The decomposition principle involves three distinct ideas:

- Those who do not use a cost element should not be charged for it.
- Everyone who uses a given cost element should be charged equally for it.
- The results of different cost allocations can be added.

Let's formalize these ideas.

- A player i is a **dummy** if it contributes no additional cost to any coalition.

The three aspects of the decomposition principle can be expressed as follows:

- **Dummy Player Property:** A dummy player is charged nothing.

- **Symmetry:** If two players enter the cost sharing game in a symmetrical way, then they should be charged equally.

- **Additivity:** If the cost sharing game c decomposes into the sum of two cost sharing games c' and c'', (that is if $c(S) = c'(S) + c''(S)$ for every coalition S) then the cost allocation for c is the sum of the allocations for c' and c''.

Example: There are three projects with the following cost structures.

	Capital Cost	Operation Cost	Total Cost
c(1)	0	3,000	3,000
c(2)	10,000	3,000	13,000
c(3)	10,000	0	10,000
c(12)	10,000	4,000	14,000
c(13)	10,000	3,000	13,000
c(23)	16,000	3,000	19,000
c(123)	16,000	4,000	20,000

Suppose that we implement all three projects and want to divide the total cost among the three projects. We can do that using the three principles we have seen.

- **Sharing the Capital Cost:**
  - 1 adds nothing to any coalition and therefore should not be charged;
  - 2 & 3 are symmetric players and therefore they should pay equally;
  - therefore they should share 16,000 capital cost as 0, 8000, 8000.
- **Sharing Operation Cost:**
  - 3 adds nothing to any coalition and therefore should not be charged;
  - 1 & 2 are symmetric players and therefore they should pay equally;
  - therefore they should share 4000 capital cost as 2000, 2000, 0.

- **Sharing Total Cost:** Using additivity
  - 1 pays 0+2000=2000,
  - 2 pays 8000+2000=10,000
  - 3 pays 8000+0=8000.

In general, it may not be easy to get such a direct answer. Nevertheless, that's not a problem. The following familiar solution gives exactly the same outcome:

- **Shapley value:** Given a cost sharing game, let the players join the game one at a time in some predetermined order. As each player joins, the number of players to be served increases. The player's cost contribution is its net addition to cost when it joins (i.e. the incremental cost of adding it to the group of players that has already joined). The Shapley value of a player is its average cost contribution over all possible orderings of the players.

Example (revisited): Let's find the Shapley value of the earlier example.

	Capital Cost	Operation Cost	Total Cost
c(1)	0	3,000	3,000
c(2)	10,000	3,000	13,000
c(3)	10,000	0	10,000
c(12)	10,000	4,000	14,000
c(13)	10,000	3,000	13,000
c(23)	16,000	3,000	19,000
c(123)	16,000	4,000	20,000

Ordering	1's share	2's share	3's share
123	3000	11,000	6000
132	3000	7000	10,000
213	1000	13,000	6000
231	1000	13,000	6000
312	3000	7000	10,000
321	1000	9000	10,000
Total	12,000	60,000	48,000
Average	2000	10,000	8000

Example:

$$\text{cost(A)} = 11 \quad \text{cost(AB)} = 15$$

$$\text{cost(B)} = 7 \quad \text{cost(AC)} = 14 \quad \text{cost(ABC)} = 20$$

$$\text{cost(C)} = 8 \quad \text{cost(BC)} = 13$$

Ordering	A's share	B's share	C's share
ABC			
ACB			
BAC			
BCA			
CAB			
CBA			
Total	50	35	35
Average	$8\frac{1}{3}$	$5\frac{5}{6}$	$5\frac{5}{6}$

- **Theorem:** The Shapley value is the only cost sharing rule that satisfies the dummy player property, symmetry, and additivity.
- If a cost sharing game decomposes into distinct cost elements (like in the power line example), the Shapley value and the decomposition principle give the same allocation. In all such situations Shapley value is in the core.
- Shapley value may not be in the core in general. For example in the last example A & C pay a total of $8\frac{1}{3} + 5\frac{5}{6} = 14\frac{1}{6}$ and yet c(AC)=14.