Equity, Equality, Proportionality

- **Aristotle's Equity Principle:** Proportionality is a very prominent norm of distributive justice.
  - When a firm goes bankrupt, all unsecured creditors in the same precedence class are repaid in proportion to the amounts they are owed.
  - If heirs are willed more than the estate is worth, the probate court usually divides it proportional to the bequests.
  - Should someone be injured by a group and it is not clear who did it, the accused parties may be assessed in proportion to the likelihood that they were responsible.
  - In 1987 several countries agreed to reduce the emissions in ozone damaging chemicals in proportion to their 1987 emission levels.

- In such situations, the parties are liable for (or entitleed to) different amounts because they differ in some respect (contribution, bequest, blame) that can be measured on a cardinal scale.
- **Proportionality** principle asserts that their shares should be in poportion to their differences.
- Two conditions should be met for proportionality to work:
  - The good must be divisible.
  - Each claimant's entitlement (or liability) must be expressable in common metric.

  These are restrictive conditions. However even when they are met, proportionality may not be the only reasonable principle.

- Let's look at the **Talmudic Law of Contracts**. Consider the following (2000 year old) problem:

  > "Two hold a garment; one claims it all, the other claims the half. What is an equitable division of the garment?"

  - Aristotle's proportionality: 1st claimant 2/3, 2nd claimant 1/3.
  - Solution in the Talmud: 1st claimant 3/4, 2nd claimant 1/4.

    Logic of this solution: The dispute is over only half of the garment. Claimants share the disputed part and the first claimant receives the undisputed part.

- **Claims Problem:** Several individuals have claims on a common asset, and the claims exceed the amount available. (Here the asset is perfectly divisible.)
- A **solution to a claims problem** is a division of the total amount among various claimants, such that no one receives more than his/her claim or less than zero.
- Two cases are important to distinguish:
  - **Voluntary Claims:** When claims are created by voluntary actions. Incentive issued may be cruial here (for example the choice of bankruptcy rule may effect the investment decisions).
  - **Involuntary Claims:** Involves no choice or effort on part of the claimant. They arise from gift, inheritence, etc. Here incentives are no issue.

We first consider the second case and avoid the incentives issues.

The Contested Garment Rule

Example:

Claim 1:	\$200,000
Claim 2:	\$300,000
Total:	\$300,000

What is an equitable allocation?

- Proportionality: \$120,000 & 180,000
- Talmudic Law of Contracts: The second claimant has an exclusive claim of \$100,000 whereas the first one does not. They should share \$200,000 equally and the first claimant should receive an additional \$100,000.

  Hence the allocation is \$100,000 & \$200,000.

The Contested Garment Rule: Let two individuals have claims against a common asset, where the sum of the claims exceed (or equal to) the total amount. Each claimant's **uncontested portion** is the amount left over after the other claimant has been paid in full in case that claim is less than the total, and zero otherwise. The **contested garment rule** gives each claimant his/her uncontested portion plus one-half of the excess over and above the sum of the uncontested portions. That is:

a_0: total amount	c_1: Claim 1
	c_2: Claim 2
m_1: 1's uncontested amount	a_1: 1's allocation
m_2: 2's uncontested amount	a_2: 2's allocation

$$m_1 = \max\{a_0 - c_2, 0\} \quad a_1 = m_1 + (a_0 - m_1 - m_2)/2$$

$$m_2 = \max\{a_0 - c_1, 0\} \quad a_2 = m_2 + (a_0 - m_1 - m_2)/2$$

Example: Let Claim 1 = \$200,000 and Claim 2 = \$300,000. Let's vary the size of the estate (i.e. a_0) between \$0 and \$500,000. Then the contested garment rule allocates the estate as follows:

- When the total is too small uncontested parts are 0, all estate is contested, and hence shared equally.
- When the total is too large then we have the following:

  $$m_1 = a_0 - c_2 \quad m_2 = a_0 - c_1$$

  and therefore

  $$a_1 = c_1 + (a_0 - c_1 - c_2)/2 \quad a_2 = c_2 + (a_0 - c_1 - c_2)/2$$

  and hence the loss is share equally.
- For intermediate values (estate between 200 and 300) Claimant 1 gets 100 (i.e. 50% of the claim) and Claimant 2 gets the rest.

- **General Rule:**
  - Equal amounts if the total is less than the smallest claim.
  - Equal loss if the total is more than the largest claim.
  - Half his/her claim to smallest claimant and rest to other in all other situations.
- Contested garment rule takes the average of the following two situations:
  - Claimant 1 arrives first and is paid full amount if his/her claim is less than or equal to total, and Claimant 2 arrives second and is paid whatever is remaining.
  - Claimant 2 arrives first and is paid full amount if his/her claim is less than or equal to total, and Claimant 1 arrives second and is paid whatever is remaining.

  This is called the **run-on-the-bank** procedure.

Example:

Claim 1 = 200, Claim 2 = 300, Total = 300

Ordering	1's portion	2's portion
1-2	200	100
2-1	0	300
Average	100	200

The Shapley Value

Question: How can we extend the contested garment rule to more than 2 individuals?

We can generalize the run on the bank procedure as follows: Compute the expected payment that each claimant receive if they run to the bank (or courts) assuming that all lineups are equally likely.

Example: Claims for A,B,C are 100, 200, 300 and the total is 400.

Ordering	A's portion	B's portion	C's portion
ABC	100	200	100
ACB	100	0	300
BAC	100	200	100
BCA	0	200	200
CAB	100	0	300
CBA	0	100	300
Total	400	700	1300
Average	$66\frac{2}{3}$	$116\frac{2}{3}$	$216\frac{2}{3}$

Shapley Value: Lineup the claimants in some arbitrary order. Beginning at the front of the line, pay each claimant in full until the funds are exhausted. The Shapley value is the average payment to each claimant over all possible orderings.

Example: Claims for A,B,C are 100, 150, 250 and the total is 220. Let's find the Shapley value.

Ordering	A's portion	B's portion	C's portion
ABC			
ACB			
BAC			
BCA			
CAB			
CBA			
Total	270	420	630
Average	45	70	105

Now suppose C leaves with her allocation. Then in the remaining problem there are two agents A, B with claims 100, 150 and the total is 220-105=115. Let's find the Shapley value of the reduced problem:

Ordering	A's portion	B's portion
AB	100	15
BA	0	115
Total	100	130
Average	50	65

- Hence the Shapley value is not consistent!
- **Question:** Can we extend the contested garment solution in a consistent way?

The answer can be found in the Talmud. Consider the following example from the Talmud:

		A's share ($c_A = 100$)	B's share ($c_B = 200$)	C's share ($c_C = 300$)
	100	$33\frac{1}{3}$	$33\frac{1}{3}$	$33\frac{1}{3}$
Total	200	50	75	75
	300	50	100	150

- **Question:** What is the general principle here?
  - The first case divides the estate equally.
  - The third case divides it proportionally.
  - The second case is something in between.

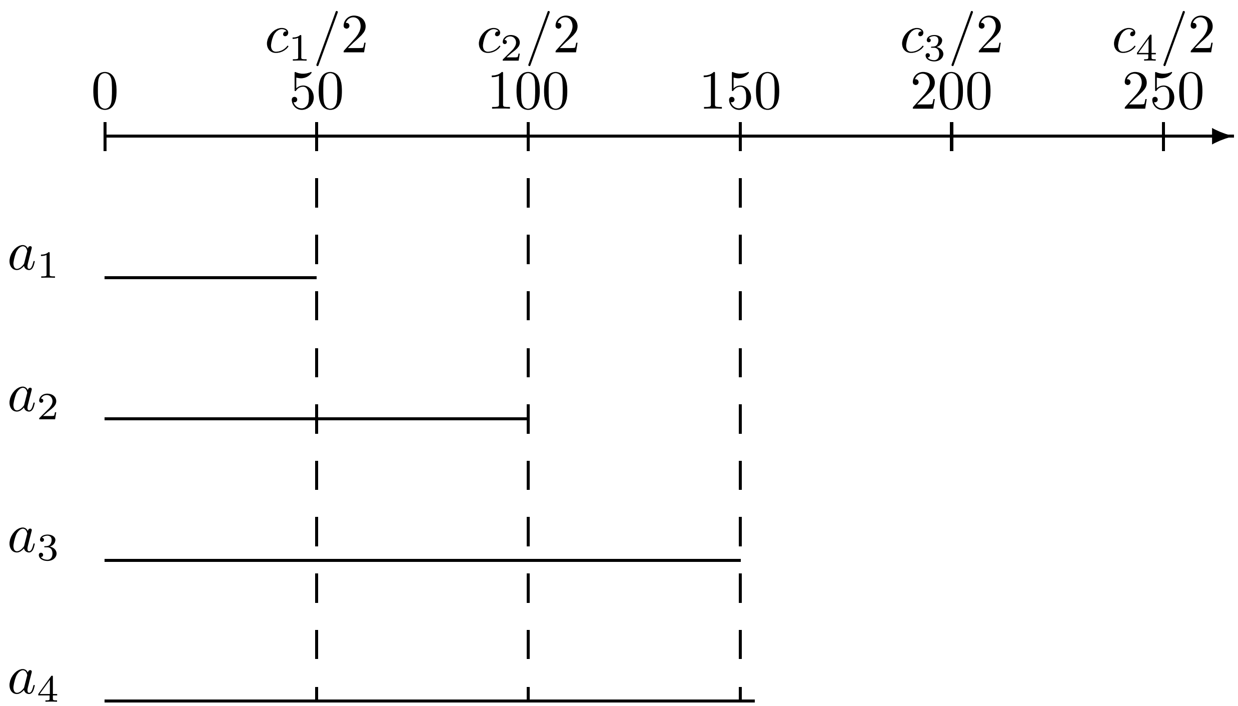
  Answer: They are all applications of the contested garment rule. Here in all cases, every pair of claimants divide the amount they are alloted according to the contested garment rule.

- **Consistency with the contested garment rule:** An allocation among a group of claimants is consistent with the contested garment rule if every two claimants share the total alloted to them according to the contested garment rule.

Talmudic solution: Order the claims from the smallest c_1 to the largest c_n. We have two cases:

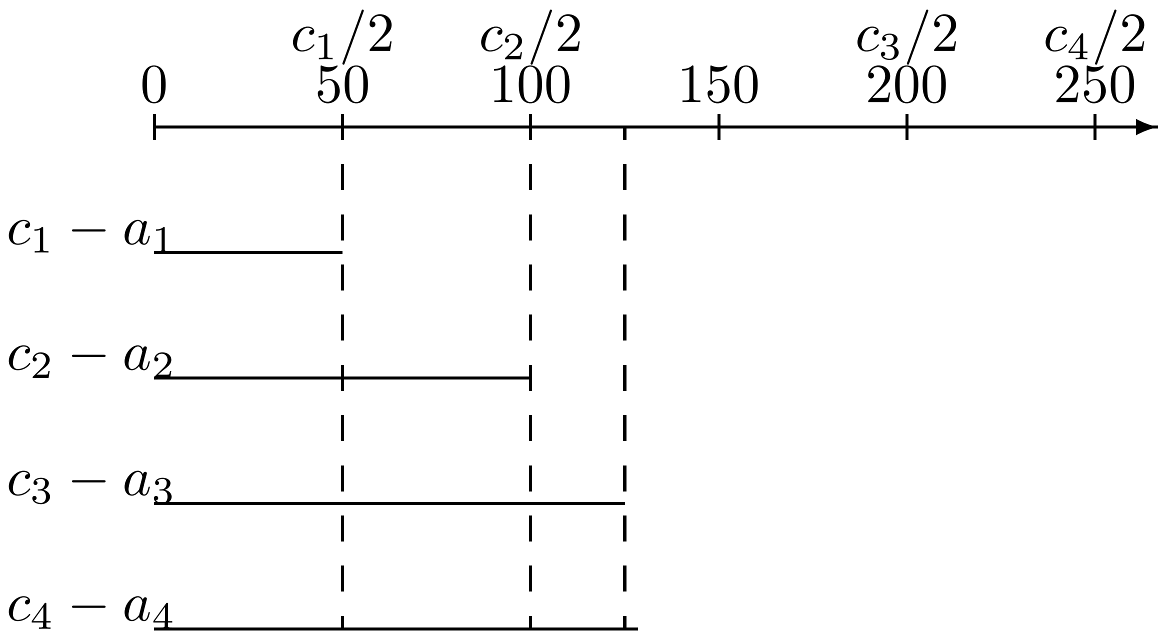
- *Case 1*: $\sum c_i/2 \geq a_0$ (Total is less than half of the claims). We'll distribute the asset in small increments.
  - Divide the first increment equally. Continue in a similar way until either the first claimant receives half of his/her claim ($c_1/2$) or the asset runs out, whichever happens first.
  - After this, divide each additional increment equally among claimants 2 through n until the second claimant receives the halfway mark ($c_2/2$) or the asset runs out.
  - Then divide the next increment equally among claimants 3 through n, and so on so forth.
- *Case 2*: $\sum c_i/2 \leq a_0$ (Total is more than half of the claims). In this case we divide the *total deficit* (i.e. $\sum c_i - a_0$) between the agents as in Case 1.

Example: $c_1 = 100$, $c_2 = 200$, $c_3 = 400$, $c_4 = 500$, $a_0 = 450$. $\sum c_i = 1200$. Therefore $\sum c_i/2 = 600 > 450 = a_0$ and hence we are in Case 1.



Therefore $a_1 = 50$, $a_2 = 100$, $a_3 = a_4 = 150$.

Example: $c_1 = 100$, $c_2 = 200$, $c_3 = 400$, $c_4 = 500$, $a_0 = 800$.
$\sum c_i = 1200$. Therefore $\sum c_i/2 = 600 < 800 = a_0$ and hence we are in Case 2. Note that we should allocate a deficit of 1200-800=400.



- $c_1 - a_1 = 50 \Rightarrow a_1 = 50$
- $c_2 - a_2 = 100 \Rightarrow a_2 = 100$
- $c_3 - a_3 = 125 \Rightarrow a_3 = 275$
- $c_4 - a_4 = 125 \Rightarrow a_4 = 375$

- **Theorem:** The Talmudic solution is the only solution to the claims problems that is consistent with the contested garment rule.

- The following solution is also from the Talmud:

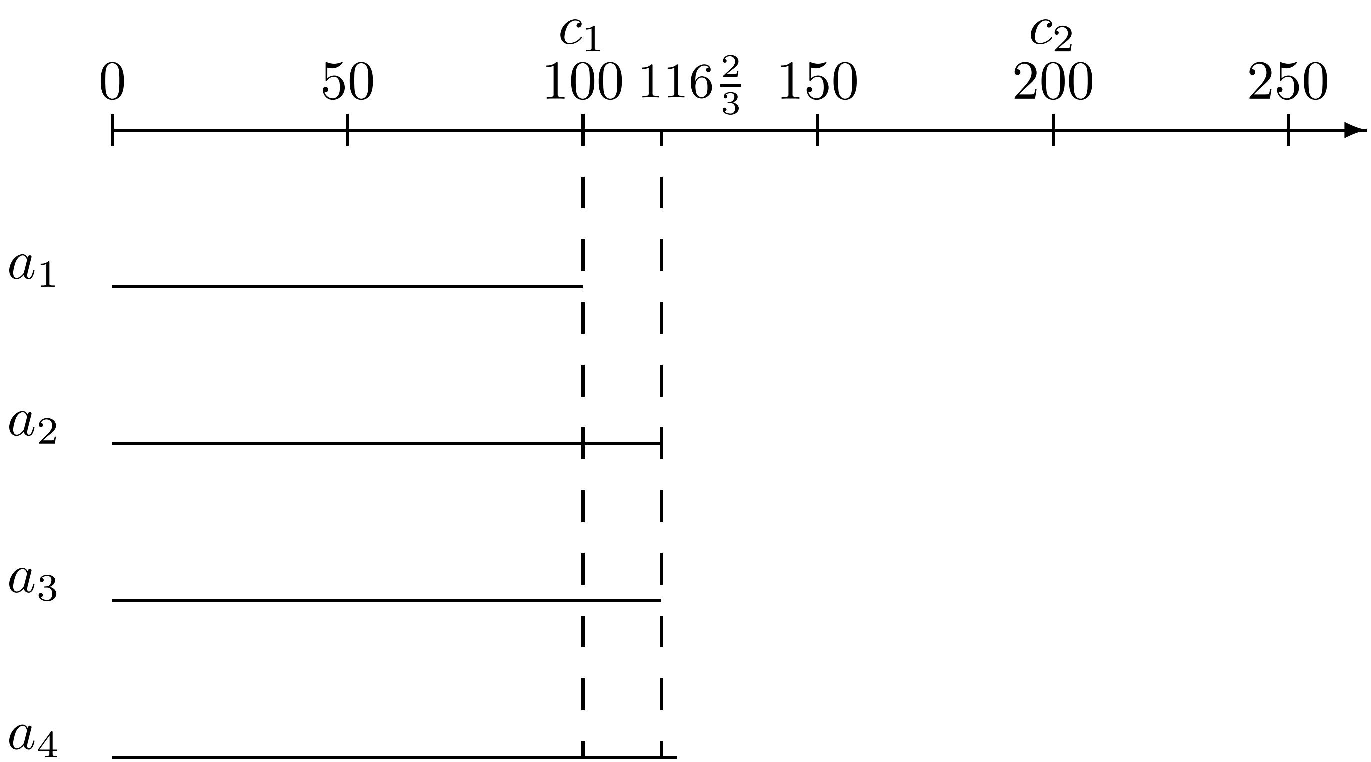
  Maimonides' Rule: Give an equal amount to every claimant or the full claim, whichever is smaller.

  We can alternatively describe it in a similar way to the Talmudic solution:

  Order the claims from the smallest c_1 to the largest c_n. We'll distribute the asset in small increments.

- Divide the first increment equally among all claimants, and continue to divide each successive unit equally until either the first claimant receives his/her claim or the asset runs out, whichever happens first.
- After this, divide each additional increment equally among claimants 2 through n until the second claimant receives his/her claim or the asset runs out.
- Then divide the next increment equally among claimants 3 through n, and so on so forth.

Example: $c_1 = 100$, $c_2 = 200$, $c_3 = 400$, $c_4 = 500$, $a_0 = 450$.



Therefore $a_1 = 100$, $a_2 = a_3 = a_4 = 116\frac{2}{3}$.

Incentive Effects

- If claims are generated by voluntary actions then incentive considerations may be an important issue in chosing a solution to claims problems.
- Suppose, for example, that Maimonides' rule were used: that is, every creditor would receive an equal amount of money up to the full amount he/she is owed. Clearly this rule might lead to serious distortions in investment behavior, because it would always be safest to be the smallest creditor.
- It would also create an incentive for creditors to find bogus partners.

- A solution to claims problems is **collusion-proof** if consolidating the claims of several individuals into one claim does not change the total amount that these claimants receive.
- A solution to claims problems is **impartial** if the allocation depends only on individuals' claims and the total amount to be distributed.
- **Theorem:** The proportional rule is the only solution to claims problems that is impartial and collusion-proof.
- This result does not say that proportional rule is completely immune from manipulation. For example, if the claims merely represent assertions by the claimants about how much they deserve, then under the proportional rule it is clearly desirable to inflate the claims.