Equity as Near as May Be

- An *apportionment problem* arises whenever a set of similar, indivisible objects must be distributed among a group of claimants in proportion to their claims.
- Here the equitable ideal is not in doubt, but it cannot be achieved because the goods are indivisible.
- One may find several applications but we will motivate and analyze this topic mainly by the distribution of legislative seats.
- An **apportionment problem** involves a group of states with given populations, and a whole number a_0 of seats to distribute among them. The population of state i will be denoted by p_i.
- An **apportionment** assigns a whole number of seats a_i to each state i, the sum being a_0.

- **Quota:** The *quota* of a state is the fraction that the state's population represents of the total population, multiplied by the total number of seats. That is:

$$q_i = \frac{p_i}{\sum_j p_j} \times a_0$$

- Typically quotas will not be whole numbers. The problem is to find a solution in whole numbers, that is "as nearly proportional to the population as possible."

- **Example**

Total Seats = 21

State	Population	Quota
A	7,270,000	14.24
B	1,230,000	2.41
C	2,220,000	4.35
Total	10,720,000	21.00

A should receive at least 14 seats, B at least 2 seats, and C at least 4 seats. Who should get the 21st seat? There has been several suggestions:

- **(Alexander) Hamilton's Method:**
  - First give each state the integer part of its quota.
  - If any seats remain, give one each to the states with highest fractional remainder.

  So in the example B gets the additional seat.

- **(Thomas) Jefferson's Method:**
  - Choose a common divisor representing the target number of persons per each seat. (A natural starting point is the total population divided by the total seats.)
  - Divide each state's population with this divisor to obtain quotients, and give each state the whole number in its quotient.
  - If the total number of seats alloted by this process is too large increase the divisor; if the total is too small decrease the divisor until a value is found that apportions the correct number of seats.

- **Example continued:** A natural initial common divisor is $10,720,000/21 = 510,476$. This yields,

  $A \to 14.24$,

  $B \to 2.41$, and

  $C \to 4.35$

  (i.e. the quotas). The total is 20. Therefore a smaller divisor should be selected.

  Select 484,000. Then the quotients are

  $A \to 15.02$,

  $B \to 2.54$, and

  $C \to 4.59$.

  The total of the whole parts in the quotients is $15 + 2 + 4 = 21$ and therefore we are done.

- Here note that Jefferson's method gives the 21st seat to A although it has the lowest fractional remainder in the quota.

- Indeed this method can yield very bizzare results: Suppose there were 22 seats. In this case the quotas are $q_A = 14.92$, $q_B = 2.52$, and $q_C = 4.56$. But Jefferson's method gives A 16 seats, B 2 seats and C 4 seats. That is, A gets more than one whole seat in excess of its quota!

- Jefferson's method systematically favors the large states! The reason is the following: By dropping the fractional part, the large state gives up only a small part of its entitlement whereas a small state may give up a major part.

  Example: Suppose there are two states A and B and the quotas are $q_A = 40.5$ and $q_B = 1.5$. Then the fraction is only 1.2% for state A but 33.3% for state B.

- In general dropping the fractional part means that the per capita representation of a small state tends to be marked down a larger percentage than a larger state.

- **Bias:** An apportionment method is **unbiased**, if over many apportionments, the difference between each state's average allotment and its average quota is approximately zero.

- **Example:** (New York in 21 cencuses)

	Average Seats	Average Quota
Hamilton's method	34.666	34.703
Jefferson'e method	35.952	34.703

  New York received 3.5% more on average under Jefferson's method.

- Jefferson's method is used from 1790's through 1830's. Based on the bias in Jefferson's method there was a lot of concern after the 1830 cencus. That resulted in the following two proposals:

- **(John Quincy) Adams' Method:** Follow the general outlines of Jeffersons method; but instead of dropping the fractional part, round it up to the next largest integer. That is,
  - Choose a common divisor representing the target number of persons per each seat.
  - Divide each state's population with this divisor to obtain quotients, and round the fractional part of each state's quotient upto the next whole number.
  - If the total number of seats alloted by this process is too large increase the divisor; if the total is too small decrease the divisor until a value is found that apportions the correct number of seats.
- The political logic of this proposal is clear: it gives an advantage to small states.

- **Webster's Method:**
  - Choose a common divisor representing the target number of persons per each seat.
  - Divide each state's population with this divisor to obtain quotients, and round the fractional part of each state's quotient to the nearest whole number.
  - If the total number of seats alloted by this process is too large increase the divisor; if the total is too small decrease the divisor until a value is found that apportions the correct number of seats.
- Webster's method was adopted in 1840 census and was used until 1850. After 1850 a variant of Hamilton's method was adopted.

What is the Most Natural Solution if There are Only Two States?

- **Example**

Total Seats = 10

State	Population	Quota
A	8,200,000	8.2
B	1,800,000	1.8
Total	10,000,000	10.00

Jefferson's method: A: 9 seats B: 1 seat

Hamilton's method: A: 8 seats B: 2 seat

Webster's method: A: 8 seats B: 2 seat

Indeed, for every two-state problem, the methods of Webster and Hamilton yield the same apportionment.

- **Standard two-state solution:** The standard two-state solution gives to each state the number of seats that is closest to its quota.
- The method's of Hamilton and Webster are two different ways of extending this idea to cases with more than two states. Webster's method, however, is the more satisfactory of the two approaches:

Example revisited:

Total Seats = 21

State	Population	Quota
A	7,270,000	14.24
B	1,230,000	2.41
C	2,220,000	4.35
Total	10,720,000	21.00

Hamilton's solution gives A 14 seats, B 3 seats, and C 4 seats. States A and B together receive 17 seats. Suppose there was a total of 17 seats and only 2 states A and B. In this case the quotas are $q_A = 14.54$ and $q_B = 2.46$. Therefore A should receive 15 seats and B should receive 2 seats. This is inconsistent!

- **Consistency with the standard two-state solution:** An apportionment method is consistent with the two-state solution if every pair of states divide the number of seats allotted to them according to the standard two-state solution.
- **Theorem**: Webster's method is the only apportionment method that is consistent with the standard two-state solution.

The Alabama Paradox

During the 19th century the number of members in House was increased every decade to accomodate new states and growing population. This process revealed another deficiency in Hamilton's method. Following 1880 cencus, the chief clerk of the cencus office computed apportionments using Hamilton's method for all house sizes between 275 and 350: Alabama was alloted 8 representatives out of a total 299, but only 7 when the total increased to 300!

- **Alabama Paradox:** A method suffers from the Alabama paradox if there is a situation in which the total number of seats increase, all populations remain fixed, and some state receives strictly fewer seats than before.

- **Example revisited:**

State	Population	Quota (21 seats)	Quota (22 seats)
A	7,270,000	14.24	14.92
B	1,230,000	2.41	2.52
C	2,220,000	4.35	4.56
Total	10,720,000	21.00	22.00

Therefore under Hamilton's method:

– For 21 seats: $A \to 14$, $B \to 3$, $C \to 4$

– For 22 seats: $A \to 15$, $B \to 2$, $C \to 5$

Hamilton's method (Maine 1900)

House size:	350-382	383-385	386	387-388	389-390	391-400
Allotment:	3	4	3	4	3	4

- Indeed Alabama Paradox is peculiar to Hamilton's method. It cannot happen under any divisor rule (i.e., Jefferson's, Adams', or Webster's).
- In 1911, Webster's method is adopted again. But not for long.

The Population Paradox

Example: Consider the following allotments by the Hamilton's method:

Total Seats = 7

State	Population	Quota	Allotment
A	752	5.013	5
B	101	0.673	1
C	99	0.660	1
D	98	0.653	0

Total Seats = 7

State	Population	Quota	Allotment
A	753	3.984	4
B	377	1.995	2
C	96	0.508	0
D	97	0.513	1

In this example D lost population and A gained. And yet D increased seats in expense of A.

- **Population paradox:** A method exhibits the population paradox if a state that loses population gains a seat at the expense of a state that gains population.

- Divisor's methods avoid the population paradox.

- **Hill's Method:**
  - Choose a common divisor representing the target number of persons per each seat.
  - Divide each state's population with this divisor to obtain quotients.
  - Round the quotient down if it is less than the geometric mean of the two nearest whole numbers, and round it up otherwise.
  - If the total number of seats alloted by this process is too large increase the divisor; if it is too small decrease the divisor until a value is found that apportions the correct number of seats.

- Hill's method allocates the seats so that no transfer of a seat between any two states reduces the percentage difference in percapita representation between them.
- Hill's method is used after 1941 until today.
- Hill's method slightly favors small states.
- Hill's method avoids Alabama and population paradoxes but it is not consistent with the two-state standard solution.

Overall Performance

	Hamilton	Jefferson	Adams	Webster	Hill
Bias	$\surd$	Poor	Poor	$\surd$	Poor
Cons.	Poor	Poor	Poor	$\surd$	Poor
A. prdx	Poor	$\surd$	$\surd$	$\surd$	$\surd$
P. prdx	Poor	$\surd$	$\surd$	$\surd$	$\surd$