The Importance of Irrelevance of Rejected Contracts in Matching under Weakened Substitutes Conditions

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Abstract

We show that Hatfield and Kojima (2010) inherits a critical ambiguity from its predecessor Hatfield and Milgrom (2005), and clearing this ambiguity has strong implications for the paper. Of the two potential remedies, the first one results in the failure of all theorems except one in the absence of an additional *irrelevance of rejected contracts* (IRC) condition, whereas the second remedy eliminates the transparency of the results, reduces the scope of the model, and contradicts authors' interpretation of the nature of their contributions. Fortunately all results are restored when IRC is explicitly assumed under the first remedy.

1 Introduction

Formulation and analysis of *matching with contracts* model (Hatfield and Milgrom 2005) is widely considered as one of the most important developments of the last twenty years in theory of matching markets.[1] This model embeds Gale and Shapley (1962) *two-sided matching* model and Kelso and Crawford (1982) *labor market model*,[2] among others. A substitutes condition that plays a key role in the analysis of Hatfield and Milgrom (2005), also induces a strong isomorphism between matching with contracts model and Kelso and Crawford (1982) labor market model (Echenique 2012). This isomorphism is considered to be a highly negative result since it reduces the scope of Hatfield and Milgrom (2005) to that of Kelso and Crawford (1982). Fortunately this restrictive "equivalence" between the two models breaks under two weaker conditions, *bilateral substitutes* and *unilateral substitutes*, introduced by Hatfield and Kojima

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[1]See also Adachi (2000), Fleiner (2003), and Echenique and Oviedo (2004).

[2]Kelso and Crawford (1982) builds on Crawford and Knoer (1981).

(2010). The significance of these weaker substitutes conditions was further increased when Sönmez and Switzer (2011) - Sönmez (2011) introduced a brand new market design application of matching with contracts, *cadet-branch matching*, which satisfies the unilateral substitutes condition but not the substitutes condition.

In this paper we show that Hatfield and Kojima (2010) inherits a critical ambiguity from its predecessor Hatfield and Milgrom (2005), and clearing this ambiguity has strong implications for the paper. There are two possible remedies to resolve this ambiguity. Of the two remedies, the first (and scientifically more sound) one results in the failure of six of the seven theorems in the absence of an additional *irrelevance of rejected contracts* (IRC) condition, whereas the second remedy eliminates the transparency of the results, reduces the scope of the model, and contradicts authors' interpretation of the role of the weaker substitutes conditions. Fortunately all results are restored when IRC is explicitly assumed under the first remedy.[3]

Since Hatfield and Kojima (2010) will likely play an important role in further applications of market design, it is important to remove the inconsistency in the model. Fortunately most market design applications of matching with contracts, including the above described cadet-branch matching, satisfy IRC, and as such they are shielded from our criticism.

2 Matching with Contracts

We mostly follow the notation of Hatfield and Milgrom (2005) and Hatfield and Kojima (2010). Since the purpose of this paper is presenting the implications of a major inconsistency in Hatfield and Kojima (2010), our presentation will also closely follow theirs.

There are finite sets D and H of doctors and hospitals, and a finite set X of contracts. Each contract $x \in X$ is associated with one doctor $x_D \in D$ and one hospital $x_H \in H$. Given a set of contracts $Y \subseteq X$, let Y_D denote the set of doctors who has contracts in Y. That is, $Y_D = \{d \in D | \; \exists y \in Y \text{ s.t. } y_D = d\}$. Each doctor $d \in D$ can sign at most one contract and his null contract where he signs no contract is denoted by $\emptyset_d$. A set of contracts $X' \subseteq X$ is an **allocation** if each doctor is associated with at most one contract under X'.

For each doctor $d \in D$, $\succ_d$ is a strict preference relation on his contracts $\big(\{x \in X | \; x_D = d\} \cup \{\emptyset_d\}\big)$. A contract is **acceptable** by doctor d if it is at least as good as the null contract $\emptyset_d$, and **unacceptable** by doctor d if it is worse that the null contract $\emptyset_d$. For each doctor $d \in D$ and a set of contracts $X' \subseteq X$, the **chosen set** $C_d(X')$ of doctor d is defined as

$$C_d(X') = \max_{\succ_d} \Big(\{x \in X' | \; x_D = d\} \cup \{\emptyset_d\}\Big).$$

For a given set of contracts $X' \subseteq X$, define $C_D(X') = \bigcup_{d \in D} C_d(X')$.

[3] In a companion note Aygün and Sönmez (2012), we show that IRC also resolves the complications in Hatfield and Milgrom (2005).

Given a hospital $h \in H$, define $X_h = \{x \in X |\ x_H = h\}$ to be the set of his non-empty contracts in X. Each hospital $h \in H$ can sign multiple contracts and has preferences $\succ_h$ on

$$\Big\{Y \subseteq X_h |\ y, y' \in Y \text{ and } y \neq y' \implies y_D \neq y'_D\Big\}.$$

Unlike doctor preferences, hospital preferences are not assumed to be strict in Hatfield and Kojima (2010). This point, which may initially seem to be a detail, will prove to be very important. For any $X' \subseteq X$, Hatfield and Kojima (2010) define the **chosen set** $C_h(X')$ of hospital h is as

$$C_h(X') = \max_{\succ_h} \Big\{Y \subseteq X' \cap X_h |\ y, y' \in Y \text{ and } y \neq y' \implies y_D \neq y'_D\Big\}.$$

Observe that the above definition of $C_h(X')$ may include more than one set of contracts unless hospital preferences are also assumed to be strict. Since choice sets are assumed to be singletons throughout the analysis in Hatfield and Kojima (2010), this definition is not well-defined. There are two possible remedies for this inconsistency. Either hospital preferences shall be assumed to be strict (as in the case of the doctors), or $C_h(X')$ shall be given as a *selection* from

$$\max_{\succ_h} \Big\{Y \subseteq X' \cap X_h |\ y, y' \in Y \text{ and } y \neq y' \implies y_D \neq y'_D\Big\}.$$

As it turns out, each remedy introduces its own complications to the analysis of Hatfield and Kojima (2010). However, we will argue that the complications associated with the latter are considerably easier to resolve, and as such, we will for now allow indifferences in hospital preferences and assume that

$$C_h(X') \in \max_{\succ_h} \Big\{Y \subseteq X' \cap X_h |\ y, y' \in Y \text{ and } y \neq y' \implies y_D \neq y'_D\Big\}.$$

For a given hospital $h \in H$, we refer the function that maps each set of contracts to a chosen set as the **choice function** of hospital h. For a given set of contracts $X' \subseteq X$, define $C_H(X') = \bigcup_{h \in H} C_h(X')$.

An important advantage of this modeling choice is that, it introduces no *a priori* constraints on the structure of chosen sets. That is because, the preference relation where a hospital h is indifferent between all subsets of X_h is consistent with any selection of chosen sets. Thereby this modeling choice is equivalent to considering hospital choice functions to be primitives of the model. In contrast, if one adopts the first remedy assuming hospitals have strict preferences, that would introduce constraints on the structure of hospital choice functions including but not limited to a version of the *strong axiom of revealed preference*, and as such, the entire analysis would be superimposed on the implied structure, inconsistent with the authors' interpretation of the results in Hatfield and Kojima (2010). We will return to this important issue in Section 6, but for now, we adopt the first remedy and thereby assume that there is no a priori structure on hospital choice functions.

3 Stable Contracts: Substitutes, IRC, and LAD

Stability axiom plays a central role in analysis of two-sided matching models, and it is extended to matching with contracts as follows:

Definition 1 *A set of contracts $X' \subseteq X$ is a* ***stable allocation*** *(or a* ***stable set of contracts****) if*

1. $C_D(X') = C_H(X') = X'$, *and*
2. *there exists no hospital $h \in H$ and set of contracts $X'' \neq C_h(X')$ such that*

$$X'' = C_h(X' \cup X'') \subseteq C_D(X' \cup X'').$$

When the first condition fails, the allocation X' fails **individual rationality** and there is a blocking doctor or a hospital. When the second condition fails, there is a blocking coalition made of an hospital h and a subset of doctors $\{x_D\}_{x \in X''}$. In this case we say that X'' **blocks** X' .

Hatfield and Milgrom (2005) claim that the set of of stable allocations is always non-empty under the following condition:

Definition 2 *Contracts are* ***substitutes*** *for hospital h if there do not exist a set of contracts $Y \subset X$ and a pair of contracts $x, z \in X \setminus Y$ such that*

$$z \notin C_h(Y \cup \{z\}) \text{ and } z \in C_h(Y \cup \{x, z\}).$$

In a companion note, Aygün and Sönmez (2012) show that the substitutes condition alone is not sufficient for the existence of a stable allocation, and the inconsistency emanates from the implicit assumption of the following additional condition in key proofs of Hatfield and Milgrom (2010).

Definition 3 *Contracts satisfy the* ***irrelevance of rejected contracts (IRC)*** *for hospital h if*

$$\forall Y \subset X, \forall z \in X \setminus Y \qquad z \notin C_h(Y \cup \{z\}) \implies C_h(Y) = C_h(Y \cup \{z\}).$$

This condition simply requires that, the removal of rejected contracts shall not affect chosen sets.[4] It turns out that, results of Hatfield and Milgrom (2005) are restored once IRC is assumed throughout their analysis.

For some of the stronger results in Hatfield and Milgrom (2005), the following condition is assumed in addition to the substitutes condition.

[4]This condition is earlier used by Blair (1988) in the context of many-to-many matching. In an extension of Blair's results, Alkan (2002) refers it as *consistency*. More recently Echenique (2007) refers this condition as *independence of irrelevant alternatives* in the context of combinatorial choice rules.

Definition 4 *Contracts satisfy the* ***law of aggregate demand (LAD)*** *for hospital h if,*

$$\forall X', X'' \subseteq X \qquad X' \subset X'' \implies |C_h(X')| \leq |C_h(X'')|.$$

Aygün and Sönmez (2012) show that the substitutes condition together with LAD imply IRC, and thereby the results in Hatfield and Milgrom (2005) which assume LAD hold without an explicit need to assume IRC.

4 Bilateral Substitutes and Unilateral Substitutes: Counter Examples

Primary contributions of Hatfield and Kojima (2010) are (1) the intoduction of two weaker versions of the substitutes condition, and (2) the analysis of the structure of stable allocations under these weaker conditions. The weakest version of substitutes introduced in Hatfield and Kojima (2010) is the following:

Definition 5 *Contracts are* ***bilateral substitutes*** *for hospital h if for any set of contracts $Y \subset X$ and any pair of contracts $x, z \in X \setminus Y$,*

$$z \notin C_h(Y \cup \{z\}) \text{ and } z \in C_h(Y \cup \{x, z\}) \implies z_D \in Y_D \text{ or } x_D \in Y_D.$$

In Theorem 1 of Hatfield and Kojima (2010), the authors claim that bilateral substitutes is sufficient for the existence of a stable allocation. More specifically, they claim that the following **cumulative offer algorithm** (Hatfield and Milgrom 2005) always produces a stable allocation:

Step 1: One of the doctors offers her first choice contract x_1. The hospital receiving the offer, $h_1 = (x_1)_H$, holds the contract if $x_1 \in C_{h_1}\big(\{x_1\}\big)$ and rejects it otherwise. Let $A_{h_1}(1) = \{x_1\}$, and $A_h(1) = \emptyset$ for all $H \setminus \{h_1\}$.

In general, at

Step t: One of the doctors with no contract on hold offers her most preferred contract x_t that has not been rejected in earlier steps. The hospital receiving the offer, $h_t = (x_t)_H$, holds the contracts in $C_{h_t}\big(A_{h_t}(t-1) \cup \{x_t\}\big)$ and rejects the rest. Let $A_{h_t}(t) = A_{h_t}(t) \cup \{x_t\}$, and $A_h(t) = A_h(t-1)$ for all $H \setminus \{h_t\}$.

The algorithm terminates when either every doctor is matched to at least one hospital or every unmatched doctor has had all acceptable contracts rejected. Since each contract is offered at most once, the algorithm terminates in some finite Step T. The outcome of the algorithm is, $\bigcup_{h \in H} C_h\big(A_h(T)\big)$.[5]

[5] Observe that while the algorithm necessarily terminates, in principle it may pick a set of contracts which is not an allocation. That is, multiple contracts of a given doctor may be chosen by the algorithm, in the absence of additional assumptions.

Given that the original (and stronger) substitutes condition is not sufficient for the existence of a stable allocation, it is clear that Theorem 1 of Hatfield and Kojima (2010) cannot hold in the absence of additional structure. Recall that results of Hatfield and Milgrom (2005) which assume LAD in addition to the substitutes condition are accurate. It turns out that, in contrast, Theorem 1 of Hatfield and Kojima (2010) fails to hold even if LAD is assumed. The following example shows that, not only the cumulative offer algorithm may produce an unstable allocation under bilateral substitutes and LAD, but also the set of stable allocations may be empty under these conditions:

Example 1 *Consider a problem with one hospital, h, and two doctors d_1, d_2. Doctor d_1 has two contracts x, x' and doctor d_2 has two contracts y, y'. Preferences of the doctors and the choice function of the hospital are given as follows:*

$$\succ_{d_1}: \quad x \succ_{d_1} x' \succ_{d_1} \emptyset_{d_1}$$
$$\succ_{d_2}: \quad y' \succ_{d_2} y \succ_{d_2} \emptyset_{d_2}$$

$C_h(\{x\}) = \{x\}$	$C_h(\{x, x'\}) = \{x\}$	$C_h(\{x, x', y\}) = \{y\}$	$C_h(\{x, x', y, y'\}) = \{x, y\}$
$C_h(\{x'\}) = \{x'\}$	$C_h(\{x, y\}) = \{y\}$	$C_h(\{x, x', y'\}) = \{x'\}$	
$C_h(\{y\}) = \{y\}$	$C_h(\{x, y'\}) = \{y'\}$	$C_h(\{x, y, y'\}) = \{x, y\}$	
$C_h(\{y'\}) = \{y'\}$	$C_h(\{x', y\}) = \{x'\}$	$C_h(\{x', y, y'\}) = \{x'\}$	
	$C_h(\{x', y'\}) = \{x'\}$		
	$C_h(\{y, y'\}) = \{y'\}$		

It is easy to verify that contracts satisfy bilateral substitutes as well as the LAD condition for hospital h.

Consider the cumulative offer algorithm and start the sequence of offers with doctor d_1. Hospital h receives the following sequence of offers: x, y', x', y. The cumulative offer algorithm terminates when all contracts are offered, and at this point $C_h(\{x, x', y, y'\}) = \{x, y\}$. Hence the outcome is $\{x, y\}$. However the allocation $\{x, y\}$ is not stable, since hospital h blocks it: $C_h(\{x, y\}) = \{y\}$. This observation directly conflicts with the proof of Theorem 1 in Hatfield and Kojima (2010) where the authors argue that the cumulative offer algorithm always results in a stable allocation under bilateral substitutes. Indeed, not only the cumulative offer algorithm yields an unstable allocation in this example, but also the set of stable allocations is empty. Here is a list of blocking coalitions for every possible allocation in this example.

Allocation	*Blocking Coalition*	*Allocation*	*Blocking Coalition*
$\{x\}$	$\{h, d_2\}$ *via* $\{y\}$	$\{x, y\}$	$\{h\}$ *via removing* x
$\{x'\}$	$\{h, d_1\}$ *via* $\{x\}$	$\{x, y'\}$	$\{h\}$ *via removing* x
$\{y\}$	$\{h, d_1\}$ *via* $\{x'\}$	$\{x', y\}$	$\{h\}$ *via removing* y
$\{y'\}$	$\{h, d_1\}$ *via* $\{x'\}$	$\{x', y'\}$	$\{h\}$ *via removing* y'

Recall that the substitutes condition together with IRC guarantee the existence of a stable allocation. Indeed, the cumulative offer algorithm gives the same stable outcome as the celebrated agent-proposing deferred acceptance algorithm under these conditions. This is no longer the case when substitutes is replaced with bilateral substitutes since a hospital may hold a contract at Step t of the cumulative offer algorithm that was rejected at an earlier Step $t' < t$. Hatfield and Kojima (2010) refers this feature as *renegotiation*. In Theorem 4 of Hatfield and Kojima (2010), the authors claim that the renegotiation feature ceases to exist and the cumulative offer algorithm yields the same outcome as the agent-proposing deferred acceptance algorithm under the following version of substitutes, that is still weaker than Hatfield and Milgrom (2005) substitutes condition, but stronger than the bilateral substitutes:

Definition 6 *Contracts are* ***unilateral substitutes*** *for hospital h if for any set of contracts $Y \subset X$ and a pair of contracts $x, z \in X \setminus Y$,*

$$z \notin C_h(Y \cup \{z\}) \text{ and } z \in C_h(Y \cup \{x, z\}) \implies z_D \in Y_D.$$

The following example refutes the claim that the cumulative offer algorithm produces the same outcome as the agent-proposing deferred acceptance algorithm under the unilateral substitutes condition:

Example 2 *Consider an allocation problem with one hospital, h, and two doctors d_1, d_2. Let doctor d_1 have three contracts x, x', x'' and doctor d_2 have one contract y. Preferences of the doctors and the choice function of hospital h are given as follows:*

$$P_{d_1}: \quad x \ P_{d_1} \ x' \ P_{d_1} \ x'' \ P_{d_1} \ \emptyset_{d_1}$$
$$P_{d_2}: \quad y \ P_{d_2} \ \emptyset_{d_2}$$

$C_h(\{x\}) = \{x\}$	$C_h(\{x, x'\}) = \{x\}$	$C_h(\{x, x', x''\}) = \{x\}$	$C_h(\{x, x', x'', y\}) = \{x, y\}$
$C_h(\{x'\}) = \{x'\}$	$C_h(\{x, x''\}) = \{x\}$	$C_h(\{x, x', y\}) = \{y\}$	
$C_h(\{x''\}) = \{x''\}$	$C_h(\{x', x''\}) = \{x'\}$	$C_h(\{x, x'', y\}) = \{y\}$	
$C_h(\{y\}) = \{y\}$	$C_h(\{x, y\}) = \{y\}$	$C_h(\{x', x'', y\}) = \{y\}$	
	$C_h(\{x', y\}) = \{y\}$		
	$C_h(\{x'', y\}) = \{y\}$		

It is easy to verify that C_h satisfies unilateral substitutes as well as LAD .

Consider the cumulative offer algorithm and start the sequence of offers with doctor d_1. Hospital h receives the following sequence of offers: x, y, x', x''. The cumulative offer algorithm terminates when all contracts are offered and at this point $C_h(\{x, x', x'', y\}) = \{x, y\}$. Hence the outcome of the cumulative offer algorithm is $\{x, y\}$. First observe that, the allocation $\{x, y\}$ is not stable, since hospital h blocks it: $C_h(\{x, y\}) = \{y\}$. Moreover it differs from the outcome of the agent-proposing deferred acceptance algorithm, which can easily be obtained as $\{y\}$.

The outcome of the agent-proposing deferred acceptance algorithm is stable in the above example. However a slightly more involved example shows that a stable allocation may fail to exist even under the unilateral substitutes condition. As such, it refutes Theorem 5 in Hatfield and Kojima (2010) in the absence of additional structure.

Example 3 *Consider an allocation problem with one hospital, h, and two doctors d_1, d_2. Let doctor d_1 have three contracts x, x', x'' and doctor d_2 have two contracts y, y'. Preferences of the doctors and the choice function of hospital h are given as follows:*

$$\succ_{d_1}: \quad x \succ_{d_1} x' \succ_{d_1} x'' \succ_{d_1} \emptyset_{d_1}$$
$$\succ_{d_2}: \quad y' \succ_{d_2} y \succ_{d_2} \emptyset_{d_2}$$

$C_h(\{x\}) = \{x\}$	$C_h(\{x, x'\}) = \{x\}$	$C_h(\{x, x', x''\}) = \{x\}$	$C_h(\{x, x', x'', y\}) = \{y\}$
$C_h(\{x'\}) = \{x'\}$	$C_h(\{x, x''\}) = \{x\}$	$C_h(\{x, x', y\}) = \{y\}$	$C_h(\{x, x', x'', y'\}) = \{x\}$
$C_h(\{x''\}) = \{x''\}$	$C_h(\{x', x''\}) = \{x'\}$	$C_h(\{x, x', y'\}) = \{x\}$	$C_h(\{x, x', y, y'\}) = \{y\}$
$C_h(\{y\}) = \{y\}$	$C_h(\{x, y\}) = \{y\}$	$C_h(\{x, x'', y\}) = \{y\}$	$C_h(\{x, x'', y, y'\}) = \{y\}$
$C_h(\{y'\}) = \{y'\}$	$C_h(\{x, y'\}) = \{x\}$	$C_h(\{x, x'', y'\}) = \{x\}$	$C_h(\{x', x'', y, y'\}) = \{y\}$
	$C_h(\{x', y\}) = \{y\}$	$C_h(\{x, y, y'\}) = \{y\}$	
	$C_h(\{x', y'\}) = \{x'\}$	$C_h(\{x', x'', y\}) = \{y\}$	
	$C_h(\{x'', y\}) = \{y\}$	$C_h(\{x', x'', y'\}) = \{x'\}$	
	$C_h(\{x'', y'\}) = \{x''\}$	$C_h(\{x', y, y'\}) = \{y\}$	
	$C_h(\{y, y'\}) = \{y'\}$	$C_h(\{x'', y, y'\}) = \{y\}$	$C_h(\{x, x', x'', y, y'\}) = \{x, y\}$

It is straightforward to verify that C_h satisfies unilateral substitutes. There is no stable allocation in this economy. Here is a list of blocking coalitions for every possible allocation:

Allocation	*Blocking Coalition*	*Allocation*	*Blocking Coalition*
$\{x\}$	$\{h, d_2\}$ *via* $\{y\}$	$\{x, y\}$	$\{h\}$ *via removing* x
$\{x'\}$	$\{h, d_2\}$ *via* $\{y\}$	$\{x, y'\}$	$\{h\}$ *via removing* y
$\{x''\}$	$\{h, d_2\}$ *via* $\{y\}$	$\{x', y\}$	$\{h\}$ *via removing* x
$\{y\}$	$\{h, d_2\}$ *via* $\{y'\}$	$\{x', y'\}$	$\{h\}$ *via removing* y'
$\{y'\}$	$\{h, d_1\}$ *via* $\{x\}$	$\{x'', y\}$	$\{h\}$ *via removing* x''
		$\{x'', y'\}$	$\{h\}$ *via removing* y'

5 Recovery with Irrelevance of Rejected Contracts

As we have shown in Section 4, several results of Hatfield and Kojima (2010) fail to hold as they are stated. Of the seven theorems in the paper, the only exception for this failure is Theorem 2. Fortunately, as in the case of Hatfield and Milgrom (2005), all results are recovered once

IRC is assumed in addition to existing hypotheses. In Appendix A, we provide proofs for the modified versions of Theorems 1, 4, and 5 which follow the general flow of the original proofs and emphasizes the role of IRC. We omit the proof for the modified version of Theorem 3, since it is not directly related to the structure of stable allocations.[6] We also omit the proofs for the modified versions of Theorems 6 and 7 since their original proofs are valid, once Theorems 1, 4, and 5 are recovered, without additional need to invoke IRC. Showing that the inconsistencies in this important research program can be eliminated with an easy fix is important because bilateral substitutes and unilateral substitutes have already established themselves not only as important conditions in theoretical analysis of matching with contracts but also for its practical applications.

While these conditions might initially appear to be minor technical deviations from the substitutes condition, a recent paper by Echenique (2012) makes it clear that they differ from it in one very significant way. In a surprising result Echenique (2012) shows that, Hatfield and Milgrom (2005) matching with contracts model is isomorphic to Kelso and Crawford (1982) labor market model under the substitutes condition. He has also shown that this isomorphism breaks under bilateral substitutes.[7] Hence applications of matching with contracts that are outside the scope of Kelso and Crawford (1982) have to rely on conditions other than the substitutes condition. Sönmez and Switzer (2011) - Sönmez (2011) have recently introduced the first market design application of matching with contracts of that nature: *Cadet-branch matching* at U.S. Army programs. Both of these market design papers heavily utilize the unilateral substitutes condition, and as such it is important to emphasize that Hatfield and Kojima (2010) research program is not broken in a substantial way. We shall also emphasize that market design applications of matching with contracts, including cadet-branch matching, are shielded from our criticism, since these applications almost always satisfy IRC.

6 Strict Hospital Preferences as Primitives

We have so far argued that the preferred way to recover the results of Hatfield and Kojima (2010) is

1. maintaining the original structure that allows for weak hospital preferences over sets of contracts that name them,

2. but adjusting the original results by imposing the IRC condition on hospital choice functions throughout the analysis.

[6]IRC still needs to be independently invoked in the "if" part of the proof of the modified version of Theorem 3.

[7]The isomorphism between the two models fail to hold under unilateral substitutes as well, as shown by Sönmez and Switzer (2011).

This approach allows us to treat hospital choice functions as primitives of the model, consistent with the presentation of several of the results in Hatfield and Kojima (2010). One might be tempted instead to recover the results by assuming hospitals have strict preferences, since IRC is directly implied in this case. We will next present why this would be a poor modeling choice.

6.1 Loss of Transparency

Let's suppose each hospital h has a *strict* preference relation $\succ_h$ over

$$\Big\{Y \subseteq X_h |\; y, y' \in Y \text{ and } y \neq y' \implies y_D \neq y'_D\Big\}, \text{ and}$$

for a given set of contracts $X' \subseteq X$ its chosen set $C_h(X')$ is derived as

$$C_h(X') = \max_{\succ_h} \Big\{Y \subseteq X' \cap X_h |\; y, y' \in Y \text{ and } y \neq y' \implies y_D \neq y'_D\Big\}. \tag{1}$$

Under this modeling choice, chosen sets are derivatives of strict hospital preferences, and as such, they must be consistent with these preferences. One potential appeal of this approach is, it assures that the resulting hospital choice functions automatically satisfy the IRC condition.[8] The IRC condition, however, is not the only condition that shall be satisfied by the resulting hospital choice functions. They shall also satisfy the following condition to assure that the underlying hospital preferences are transitive.

Definition 7 *Contracts satisfy the* ***Strong Axiom of Revealed Preference (SARP)*** *for hospital h, if there exists no distinct $X^1, X^2, \ldots, X^k \subset X$ and distinct $Y^1, Y^2, \ldots, Y^k \subset X$ with $k > 1$, such that*

$$\begin{array}{lll} \forall \ell \in \{1, \ldots, k\} & Y^\ell = C_h(X^\ell), \text{ and} & \\ \forall \ell \in \{1, \ldots, k-1\} & Y^\ell \subset X^\ell \cap X^{\ell+1} \quad \text{and} & Y^k \subset X^k \cap X^1. \end{array}$$

So before the analysis even starts, there is strong *a priori* structure imposed on hospital choice functions under this approach. This is especially troubling since the key results of Hatfield and Kojima (2010) concern the impact of particular properties of hospital choice functions on sets of stable allocations or mechanisms that select stable allocations. A loose analogy here would be, trying to appreciate a picture that is drawn on top of another picture. To illustrate how this affects the interpretation of their results, let's take Theorem 1 of Hatfield and Kojima (2010). This result reads:

Result 1 *Suppose that contracts are bilateral substitutes for every hospital. Then there exists a stable allocation.*

[8] See, for example, Lemma 1 in Hatfield, Immorlica and Kominers (2012) for a short proof of this observation.

The reader, however, is expected to interpret this statement as follows:

> *Consider hospital choice functions that can be obtained from strict hospital preferences via Relation 1. In addition, suppose that contracts are bilateral substitutes for every hospital. Then there exists a stable allocation.*

As such, the exact role of bilateral substitutes in this existence result is not transparent. Perhaps the existence is "mostly" due to the underlying structure of feasible hospital choice functions which is already imposed before bilateral substitutes. Hence all results shall be interpreted in the context of an underlying structure which is not even discussed in the paper.

At the end, majority of results in Hatfield and Kojima (2010) rely on IRC no matter how the ambiguity is resolved. When hospital choice functions are treated as primitives (or alternatively when underlying hospital preferences allow for indifferences), this condition is explicitly stated in the results. When hospital preferences are strict, this condition is not only hidden in the results, but also accompanied by another implicit assumption, SARP, which has no role in any of the proofs. We next elaborate on how this redundancy reflects itself on applications of this important research program.

6.2 Reduced Scope of the Analysis

None of the results in Hatfield and Kojima (2010) rely on the SARP condition discussed in Section 6.1, and its sole purpose is assuring the existence of underlying strict preferences for the hospitals. While it is certainly important to cover choice functions that are derivatives of strict hospital preferences, *imposing* such a structure significantly reduces the scope of the analysis without any clear benefit. Indeed, Sönmez and Switzer (2011) and Sönmez (2011) have recently presented the first practical application of the unilateral substitutes condition in a brand new application of market design, cadet-branch matching, and this first application builds on choice functions that are derivatives of branch priorities that capture Army policies, and they are not derivatives of branch preferences. This and similar potential applications of Hatfield and Kojima (2010) might be left outside the scope of their paper, if hospital choice functions are *required* to be derivatives of underlying strict preferences.

6.3 Adverse Impact on Interpretation of the Results

While the substitutes condition and SARP are logically independent in the absence of other conditions, the substitutes condition together with IRC (or alternatively together with LAD) imply SARP (Aygün and Sönmez 2012). What that means is, once IRC is assured, the substitutes condition guarantee the existence of underlying strict hospital preferences. It turns out that, this result has no counterpart for bilateral substitutes or even for the stronger unilateral substitutes. In Appendix B we present an (admittedly involved) example where the bilateral substitutes condition holds along with IRC and LAD, and yet SARP fails to holds.[9] Therefore one might lose the guaranteed existence of an underlying strict hospital preference relation upon weakening the substitutes condition to either bilateral substitutes or unilateral substitutes. In other words relaxing the substitutes condition to its weaker versions may not be "free." This is in sharp contrast with the authors' interpretation of their results, and promotion of their weaker substitutes conditions. To illustrate this point, consider the following statement in page 1715:

> We have seen that the bilateral substitutes condition is a useful notion in matching with contracts in the sense that it is the weakest condition guaranteeing the existence of a stable allocation known to date.

Since bilateral substitutes guarantee existence of a stable allocation only in the presence of an underlying structure a priori imposed on hospital choice functions, the interaction of bilateral substitutes with the underlying structure is important. In the presence of IRC, the substitutes condition guarantee compatibility with the underlying structure, whereas bilateral substitutes or unilateral substitutes do not. As such, bilateral substitutes (or unilateral substitutes) can no longer be considered to be a "costless" relaxation of the substitutes condition. Observe that this issue is entirely caused by compatibility with SARP, which was never needed in entire analysis. Therefore taking strict hospital preferences as primitives of the model introduces an artificial difficulty in interpretation of the role of the weaker substitutes conditions.

7 Concluding Remarks

We presented two remedies to resolve a critical inconsistency in Hatfield and Kojima (2010). We believe the first one which essentially treats hospital choice functions as the primitives of the model is the scientifically sound remedy since it maintains the transparency of the results, increases the scope of the paper by embracing applications such as cadet-branch matching, and allows for more transparent comparisons between the roles of various substitutes conditions. It is important to emphasize that market design applications of matching with contracts almost always satisfy the IRC condition, and therefore shielded from our criticism.

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A Proofs for Modified Versions of Theorems 1, 4 and 5

We will mostly follow the general outline of the original proofs in Hatfield and Kojima (2010), so that the extensive role of IRC can be clearly observed.

Theorem 1 *Suppose that contracts are bilateral substitutes for every hospital and they satisfy IRC. Then there exists a stable allocation.*

Proof. Suppose that contracts are bilateral substitutes for every hospital and they satisfy IRC. We will show that the cumulative offer algorithm yields a stable allocation under these conditions. Cumulative offer algorithm always terminates in finite steps and produces a set of contracts since there are finite number of contracts, and no contract can be offered more than once. Let the algorithm terminate at Step T producing the set of contracts X'. We want to show that X' is a stable allocation.

We first show that X' is an allocation. To do so, we will show that no doctor can have multiple contracts in his name under X'. This is a direct implication of the following Claim which states that a hospital cannot hold at any step a contract it rejected in the previous step unless in the previous step it holds another contract of the same doctor. This does not rule out the possibility that a previously rejected contract to be held later on, but it rules out the possibility that multiple contracts of the same doctor to be on hold at any given step across all hospitals.

***Claim*:** For any $h \in H$, $z \in X$ with $z_H = h$, and $t \geq 2$,

$$z \in A_h(t-1) \setminus C_h\big(A_h(t-1)\big) \text{ and } z_D \notin \Big[C_h\big(A_h(t-1)\big)\Big]_D \implies z \notin C_h\big(A_h(t)\big).$$

Proof of the Claim: We have three cases to consider.

Case 1: Hospital h receives no offers at Step t.

This case immediately follows since $A_h(t-1) = A_h(t)$.

Case 2: Hospital h receives an offer z' from doctor z_D at Step t.

Since $z \in A_h(t-1)$, we have $z' \neq z$, and thus $A_h(t) = A_h(t-1) \cup \{z'\}$. Towards a contradiction suppose $z \in C_h\big(A_h(t)\big)$. Then $z' \notin C_h\big(A_h(t)\big)$ and hence by IRC we have $C_h\big(A_h(t)\big) = C_h\big(A_h(t-1)\big)$ contradicting $z \in C_h\big(A_h(t)\big) \setminus C_h\big(A_h(t-1)\big)$ and completing Case 2.

Case 3: Hospital h receives and offer x from doctor $x_D \neq z_D$ at Step t.

Let $Y = A_h(t-1) \setminus \big\{y \in X |\ y_D \in \{x_D, z_D\}\big\}$. Observe that $x_D, z_D \notin Y_D$. Since doctor x_D makes an offer at Step t, we have $x_D \notin \Big[C_h\big(A_h(t-1)\big)\Big]_D$; furthermore by assumption $z_D \notin \Big[C_h\big(A_h(t-1)\big)\Big]_D$. Finally by IRC, $C_h\big(A_h(t-1)\big) = C_h(Y \cup \{z\})$, and therefore $z \notin C_h(Y \cup \{z\})$, which in turn implies

$$z \notin C_h(Y \cup \{x, z\}) \tag{2}$$

by bilateral substitutes. Towards a contradiction suppose $z \in C_h\big(A_h(t)\big)$. Since $z \notin C_h\big(A_h(t-1)\big)$, that means $C_h\big(A_h(t)\big) \neq C_h\big(A_h(t-1)\big)$, which in turn implies $x \in C_h\big(A_h(t)\big)$ by IRC and $A_h(t) = A_h(t-1) \cup \{x\}$. Thus $x, z \in C_h\big(A_h(t)\big)$ which means neither doctor x_D nor doctor z_D can have another contacts in $C_h\big(A_h(t)\big)$. Therefore IRC implies $x, z \in C_h(Y \cup \{x, z\})$ contradicting Relation (2) and completing Case 3. This completes the proof of the Claim. $\square$

We will next show that allocation X' is stable. First observe that no doctor can block X' since a doctor never offers an unacceptable contract. Hence $C_D(X') = X'$. Next suppose $C_H(X') \neq X'$, and observe that $C_H(X') = \bigcup_{h \in H} C_h\big(A_h(T)\big)$ under IRC. Therefore there exists a hospital h and a contract x such that $x \in C_h(A_h(T))$ but $x \notin C_h\big(C_h(A_h(T))\big)$. This is ruled out by IRC and hence $C_H(X') = X'$.

Finally, towards a contradiction, suppose there exists a hospital h and a set of contracts $X'' \neq C_h(X')$ such that

$$X'' = C_h(X' \cup X'') \subseteq C_D(X' \cup X'').$$

Let $X'_h = \{x \in X' |\ x_H = h\}$. That is, X'_h is the subset of X' that pertains to hospital h. Observe that $X'_h = C_h\big(A_h(T)\big)$ by the mechanics of the cumulative offer algorithm. Also recall that, we have already shown $C_h(X') = X'_h$ by the above individual rationality argument. Hence

$$X'_h = C_h(X') = C_h\big(A_h(T)\big). \tag{3}$$

Since $X'' = C_h(X' \cup X'')$, we have $x_H = h$ for all $x \in X''$. Moreover since $X'' \subseteq C_D(X' \cup X'')$,

$$\forall x \in X'' \qquad x \ \succeq_{x_D} \ x'_{x_D}.$$

Therefore each contract in X'' is offered to hospital h by step T by the mechanics of the cumulative offer algorithm. Hence

$$X'' \subseteq A_h(T). \tag{4}$$

This in turn implies

$$X'' = C_h(X' \cup X'') = C_h(X'_h \cup X'') = C_h\big(C_h(X') \cup X''\big) = C_h(A_h(T)) = C_h(X')$$

contradicting $X'' \neq C_h(X')$. Here

1. the first equality holds by assumption;

2. the second equality holds by IRC since none of the contracts in $X' \setminus X'_h$ pertain to hospital h, and as such they are automatically rejected by h;

3. the third equality holds by the Relation (3);

4. the fourth equality holds by IRC together with Relations (3) and (4) since $\big(C_h(X') \cup X''\big) \subseteq A_h(T)$ and only the rejected contracts are removed between $A_h(T)$ and $\big(C_h(X') \cup X''\big)$; and

5. the last equality holds by the Relation (3).

This shows that X' is stable completing the proof. ∎

The next theorem by Hatfield and Kojima (2010) states that a contract that is rejected at any step is rejected for good under unilateral substitutes and IRC.

Theorem 4 *Suppose that contracts are unilateral substitutes for every hospital and they satisfy IRC. A contract z that is rejected by a hospital h at any step of the cumulative offer algorithm cannot be held by hospital h in any subsequent step.*

Proof. Towards a contradiction let t' be the first step a hospital h holds a contract z it previously rejected at Step $t < t'$. Since z is rejected by hospital h at Step t, either it was on hold by hospital h at Step $(t-1)$ or it was offered to hospital h at Step t. In either case no other contract of doctor z_D could be on hold by hospital h at Step $(t-1)$. But then, since z is the first contract to be held after an earlier rejection, hospital h cannot have held another contract by doctor z_D at Step t. That is,

$$z_D \notin \Big[C_h\big(A_h(t)\big)\Big]_D. \tag{5}$$

Then by IRC $z \in A_h(t) \setminus C_h\big(A_h(t)\big)$ implies

$$z \notin C_h\Big(C_h\big(A_h(t)\big) \cup \{z\}\Big), \tag{6}$$

and yet

$$z \in C_h\big(A_h(t')\big). \tag{7}$$

Since $\Big(C_h\big(A_h(t)\big) \cup \{z\}\Big) \subseteq A_h(t')$, relations (5), (6), and (7) contradict unilateral substitutes completing the proof. ∎

Theorem 5 *Suppose that contracts are unilateral substitutes for every hospital and they satisfy IRC. Then there exists a doctor-optimal stable allocation each doctor weakly prefers to any other stable allocation. The allocation that is produced by the cumulative offer algorithm is the doctor-optimal stable allocation.*

Proof. Since unilateral substitutes implies bilateral substitutes, there exists a stable allocation by Theorem 1. To prove the theorem, it suffices to show that for any stable allocation $X' \subseteq X$ and any contract $z \in X'$, contract z is not rejected by the cumulative offer algorithm. To obtain the desired contradiction, suppose not. Let t be the first step where a hospital $h = z_H$ rejects such an allocation z, and let $Y = C_h\big(A(t)\big)$. Then by IRC, $z \notin C_h\big(Y \cup \{z\}\big)$. By Theorem 4, $z_D \notin Y_D$. Since t is the first step a contract in any stable allocation is rejected, every doctor in Y_D weakly prefers their contract in Y to their contract in X' which is stable by assumption. We complete the proof via two cases each of which yields the desired contradiction:

Case 1: $z \notin C_h(Y \cup X')$. In this case hospital h blocks allocation X' together with doctors in Y_D (unless $Y_D = \emptyset$ in which case hospital h blocks X' by itself). That is, Y blocks X' contradicting its stability.

Case 2: $z \in C_h(Y \cup X')$. This case immediately gives a contradiction by unilateral substitutes since $(Y \cup \{x\}) \subseteq (Y \cup X')$, $z \notin C_h(Y \cup \{x\})$, and $z_D \notin Y_D$. ■

Hatfield and Kojima (2010) observe that, the cumulative offer algorithm overlaps with the doctor proposing deferred acceptance algorithm (for any sequence of offers). This observation is directly implied by Theorem 5.

B Omitted Example

The following example shows that, unlike for the case of the substitutes condition, SARP may be violated under bilateral substitutes even when IRC and LAD are satisfied. A more involved example which fails SARP but satisfies the unilateral substitutes condition, IRC and LAD is available upon request. That example, however, is even more involved, and relies on a total of eight contracts rather than the seven we utilized to construct Example 4.

Example 4 *Consider a problem with one hospital, h, and two doctors d_1, d_2. Doctor d_1 has three contracts x^0, x^1, x^2 and doctor d_2 has four contracts y^0, y^1, y^2, y^3. Choice function of hospital h is given in Table 1.*

In this problem contracts satisfy IRC, LAD, and bilateral substitutes, but violate SARP. Violation of SARP is due to the following "4-cycle":

$$\begin{aligned} C_h(\{x^0, y^0, y^3\}) &= \{y^3\} \\ C_h(\{x^2, y^2, y^3\}) &= \{x^2, y^2\} \\ C_h(\{x^1, x^2, y^1, y^2\}) &= \{x^1, y^1\} \\ C_h(\{x^0, x^1, y^0, y^1\}) &= \{x^0, y^0\} \end{aligned}$$

Observe that if $\succ_h$ were to denote the underlying strict preference relation for hospital h, then we would have

$$\{y^3\} \succ_h \{x^2, y^2\} \succ_h \{x^1, y^1\} \succ_h \{x^0, y^0\} \succ_h \{y^3\}$$

contradicting the transitivity of $\succ_h$.

Table 1: Choice function for hospital h. Boldface choices indicate the set of contracts that result in a violation of SARP.

Y	$C_h(Y)$	Y	$C_h(Y)$	Y	$C_h(Y)$	Y	$C_h(Y)$
$\{x^0\}$	$\{x^0\}$	$\{x^0, x^1\}$	$\{x^0\}$	$\{x^0, x^1, x^2\}$	$\{x^0\}$	$\{x^0, x^1, x^2, y^0\}$	$\{x^2, y^0\}$
$\{x^1\}$	$\{x^1\}$	$\{x^0, x^2\}$	$\{x^0\}$	$\{x^0, x^1, y^0\}$	$\{x^0, y^0\}$	$\{x^0, x^1, x^2, y^1\}$	$\{x^1, y^1\}$
$\{x^2\}$	$\{x^2\}$	$\{x^0, y^0\}$	$\{x^0, y^0\}$	$\{x^0, x^1, y^1\}$	$\{x^1, y^1\}$	$\{x^0, x^1, x^2, y^2\}$	$\{x^0, y^2\}$
$\{y^0\}$	$\{y^0\}$	$\{x^0, y^1\}$	$\{x^0, y^1\}$	$\{x^0, x^1, y^2\}$	$\{x^0, y^2\}$	$\{x^0, x^1, x^2, y^3\}$	$\{x^1, y^3\}$
$\{y^1\}$	$\{y^1\}$	$\{x^0, y^2\}$	$\{x^0, y^2\}$	$\{x^0, x^1, y^3\}$	$\{x^1, y^3\}$	$\{x^0, x^1, y^0, y^1\}$	$\boldsymbol{\{x^0, y^0\}}$
$\{y^2\}$	$\{y^2\}$	$\{x^0, y^3\}$	$\{y^3\}$	$\{x^0, x^2, y^0\}$	$\{x^2, y^0\}$	$\{x^0, x^1, y^0, y^2\}$	$\{x^0, y^2\}$
$\{y^3\}$	$\{y^3\}$	$\{x^1, x^2\}$	$\{x^1\}$	$\{x^0, x^2, y^1\}$	$\{x^0, y^1\}$	$\{x^0, x^1, y^0, y^3\}$	$\{x^1, y^3\}$
		$\{x^1, y^0\}$	$\{x^1, y^0\}$	$\{x^0, x^2, y^2\}$	$\{x^0, y^2\}$	$\{x^0, x^1, y^1, y^2\}$	$\{x^0, y^2\}$
		$\{x^1, y^1\}$	$\{x^1, y^1\}$	$\{x^0, x^2, y^3\}$	$\{y^3\}$	$\{x^0, x^1, y^1, y^3\}$	$\{x^1, y^3\}$
		$\{x^1, y^2\}$	$\{x^1, y^2\}$	$\{x^0, y^0, y^1\}$	$\{x^0, y^0\}$	$\{x^0, x^1, y^2, y^3\}$	$\{x^1, y^3\}$
		$\{x^1, y^3\}$	$\{x^1, y^3\}$	$\{x^0, y^0, y^2\}$	$\{x^0, y^2\}$	$\{x^0, x^2, y^0, y^1\}$	$\{x^2, y^0\}$
		$\{x^2, y^0\}$	$\{x^2, y^0\}$	$\{x^0, y^0, y^3\}$	$\boldsymbol{\{y^3\}}$	$\{x^0, x^2, y^0, y^2\}$	$\{x^2, y^0\}$
		$\{x^2, y^1\}$	$\{x^2, y^1\}$	$\{x^0, y^1, y^2\}$	$\{x^0, y^2\}$	$\{x^0, x^2, y^0, y^3\}$	$\{x^2, y^0\}$
		$\{x^2, y^2\}$	$\{x^2, y^2\}$	$\{x^0, y^1, y^3\}$	$\{y^3\}$	$\{x^0, x^2, y^1, y^2\}$	$\{x^0, y^2\}$
$\{x^0, x^1, x^2, y^0, y^1\}$	$\{x^2, y^0\}$	$\{x^2, y^3\}$	$\{y^3\}$	$\{x^0, y^2, y^3\}$	$\{x^0, y^2\}$	$\{x^0, x^2, y^1, y^3\}$	$\{y^3\}$
$\{x^0, x^1, x^2, y^0, y^2\}$	$\{x^2, y^0\}$	$\{y^0, y^1\}$	$\{y^0\}$	$\{x^1, x^2, y^0\}$	$\{x^2, y^0\}$	$\{x^0, x^2, y^2, y^3\}$	$\{x^0, y^2\}$
$\{x^0, x^1, x^2, y^0, y^3\}$	$\{x^1, y^3\}$	$\{y^0, y^2\}$	$\{y^2\}$	$\{x^1, x^2, y^1\}$	$\{x^1, y^1\}$	$\{x^0, x^0, y^1, y^2\}$	$\{x^0, y^2\}$
$\{x^0, x^1, x^2, y^1, y^2\}$	$\{x^0, y^2\}$	$\{y^0, y^3\}$	$\{y^3\}$	$\{x^1, x^2, y^2\}$	$\{x^1, y^2\}$	$\{x^0, y^0, y^1, y^3\}$	$\{y^3\}$
$\{x^0, x^1, x^2, y^1, y^3\}$	$\{x^1, y^3\}$	$\{y^1, y^2\}$	$\{y^2\}$	$\{x^1, x^2, y^3\}$	$\{x^1, y^3\}$	$\{x^0, y^0, y^2, y^3\}$	$\{x^0, y^2\}$
$\{x^0, x^1, x^2, y^2, y^3\}$	$\{x^1, y^3\}$	$\{y^1, y^3\}$	$\{y^3\}$	$\{x^1, y^0, y^1\}$	$\{x^1, y^1\}$	$\{x^0, y^1, y^2, y^3\}$	$\{x^0, y^2\}$
$\{x^0, x^1, y^0, y^1, y^2\}$	$\{x^0, y^2\}$	$\{y^2, y^3\}$	$\{y^2\}$	$\{x^1, y^0, y^2\}$	$\{x^1, y^0\}$	$\{x^1, x^2, y^0, y^1\}$	$\{x^2, y^0\}$
$\{x^0, x^1, y^0, y^1, y^3\}$	$\{x^1, y^3\}$			$\{x^1, y^0, y^3\}$	$\{x^1, y^3\}$	$\{x^1, x^2, y^0, y^2\}$	$\{x^2, y^0\}$
$\{x^0, x^1, y^0, y^2, y^3\}$	$\{x^1, y^3\}$			$\{x^1, y^1, y^2\}$	$\{x^1, y^1\}$	$\{x^1, x^2, y^0, y^3\}$	$\{x^1, y^3\}$
$\{x^0, x^1, y^1, y^2, y^3\}$	$\{x^1, y^3\}$			$\{x^1, y^1, y^3\}$	$\{x^1, y^3\}$	$\{x^1, x^2, y^1, y^2\}$	$\boldsymbol{\{x^1, y^1\}}$
$\{x^0, x^2, y^0, y^1, y^2\}$	$\{x^2, y^0\}$			$\{x^1, y^2, y^3\}$	$\{x^1, y^3\}$	$\{x^1, x^2, y^1, y^3\}$	$\{x^1, y^3\}$
$\{x^0, x^2, y^0, y^1, y^3\}$	$\{x^2, y^0\}$			$\{x^2, y^0, y^1\}$	$\{x^2, y^0\}$	$\{x^1, x^2, y^2, y^3\}$	$\{x^1, y^3\}$
$\{x^0, x^2, y^0, y^2, y^3\}$	$\{x^2, y^0\}$	$\{x^0, x^1, x^2, y^0, y^1, y^2\}$	$\{x^2, y^0\}$	$\{x^2, y^0, y^2\}$	$\{x^2, y^0\}$	$\{x^1, y^0, y^1, y^2\}$	$\{x^1, y^1\}$
$\{x^0, x^2, y^1, y^2, y^3\}$	$\{x^0, y^2\}$	$\{x^0, x^1, x^2, y^0, y^1, y^3\}$	$\{x^1, y^3\}$	$\{x^2, y^0, y^3\}$	$\{x^2, y^0\}$	$\{x^1, y^0, y^1, y^3\}$	$\{x^1, y^3\}$
$\{x^0, y^0, y^1, y^2, y^3\}$	$\{x^0, y^2\}$	$\{x^0, x^1, x^2, y^0, y^2, y^3\}$	$\{x^1, y^3\}$	$\{x^2, y^1, y^2\}$	$\{x^2, y^2\}$	$\{x^1, y^0, y^2, y^3\}$	$\{x^1, y^3\}$
$\{x^1, x^2, y^0, y^1, y^2\}$	$\{x^2, y^0\}$	$\{x^0, x^1, x^2, y^1, y^2, y^3\}$	$\{x^1, y^3\}$	$\{x^2, y^1, y^3\}$	$\{y^3\}$	$\{x^1, y^1, y^2, y^3\}$	$\{x^1, y^3\}$
$\{x^1, x^2, y^0, y^1, y^3\}$	$\{x^1, y^3\}$	$\{x^0, x^1, y^0, y^1, y^2, y^3\}$	$\{x^1, y^3\}$	$\{x^2, y^2, y^3\}$	$\boldsymbol{\{x^2, y^2\}}$	$\{x^2, y^0, y^1, y^2\}$	$\{x^2, y^0\}$
$\{x^1, x^2, y^0, y^2, y^3\}$	$\{x^1, y^3\}$	$\{x^0, x^2, y^0, y^1, y^2, y^3\}$	$\{x^2, y^0\}$	$\{y^0, y^1, y^2\}$	$\{y^2\}$	$\{x^2, y^0, y^1, y^3\}$	$\{x^2, y^0\}$
$\{x^1, x^2, y^1, y^2, y^3\}$	$\{x^1, y^3\}$	$\{x^1, x^2, y^0, y^1, y^2, y^3\}$	$\{x^1, y^3\}$	$\{y^0, y^1, y^3\}$	$\{y^3\}$	$\{x^2, y^0, y^2, y^3\}$	$\{x^2, y^0\}$
$\{x^1, y^0, y^1, y^2, y^3\}$	$\{x^1, y^3\}$			$\{y^0, y^2, y^3\}$	$\{y^2\}$	$\{x^2, y^1, y^2, y^3\}$	$\{x^2, y^2\}$
$\{x^2, y^0, y^1, y^2, y^3\}$	$\{x^2, y^0\}$	$\{x^0, x^1, x^2, y^0, y^1, y^2, y^3\}$	$\{x^1, y^3\}$	$\{y^1, y^2, y^3\}$	$\{y^2\}$	$\{y^0, y^1, y^2, y^3\}$	$\{y^2\}$