Matching with Contracts: The Critical Role of Irrelevance of Rejected Contracts

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Abstract

We show that an ambiguity in setting the primitives of the *matching with contracts* model by Hatfield and Milgrom (2005) has serious implications for the model. Of the two ways to clear the ambiguity, the first (and what we consider more "clean") remedy renders several of the results of the paper invalid in the absence of an additional *irrelevance of removed contracts* condition implicitly assumed throughout the paper, whereas the second remedy results in the lack of transparency in presentation of results while at the same time reducing the scope of the analysis with no clear benefit.

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1 Introduction

Matching with contracts model (Hatfield and Milgrom 2005) is widely considered as one of the most important advances of the last two decades in matching theory. This powerful model embeds Gale and Shapley (1962) *two-sided matching* model and Crawford and Knoer (1981) - Kelso and Crawford (1982) *labor market model*,[1] and it has given impetus to a flurry of theoretical research as well as new applications of market design such as *school choice with soft-caps* by Hafalir, Yenmez and Yildirim (2011) and *cadet-branch matching* by Sönmez and Switzer (2011) - Sönmez (2011). Utilizing fixed-point techniques from lattice theory, Hatfield and Milgrom (2005) analyze the set of stable allocations in their rich framework. One of the main messages of their paper is that the set of stable allocations is non-empty and it is a complete lattice under a *substitutes* condition. In this note we show that an additional *irrelevance of rejected contracts* (IRC) condition is implicitly assumed throughout their analysis, and in the absence of this condition several of their results, including the guaranteed existence of a stable allocation, fail to hold.

The complication in the paper emanates from an ambiguity in setting the primitives of the model. While hospital choices from sets of contracts are motivated by underlying hospital preferences, the resulting choice functions are treated as if they were the primitives throughout Hatfield and Milgrom (2005). If choice functions are indeed intended to be endogenously derived from given strict hospital preferences, they would automatically satisfy IRC restoring their results.[2] However this easy fix would impose additional structure on hospital choice functions, beyond that implied by IRC, to maintain the transitivity of the underlying preferences. That is, hospital choice functions would need to satisfy a *strong axiom of revealed preference* (SARP), in addition to IRC. This in turn would mean that, not only the role of the substitutes condition in entire analysis becomes more opaque being superimposed on a complicated choice structure, but also the scope of the results becomes less general since SARP is not otherwise needed in the analysis. In that sense, letting hospital choice functions to be primitives of the model (consistent with the analysis in Hatfield and Milgrom 2005), and explicitly assuming IRC throughout the analysis appears to be the superior approach to correctly interpret the results of Hatfield and Milgrom (2005).

[1] In a recent paper, Echenique (2012) shows that matching with contracts can be embedded within Kelso and Crawford (1982) under a substitutes condition, which also plays a central role in this note.

[2] Hatfield and Milgrom (2005) do not specify how hospital choices relate to an underlying preference relation.

2 Matching with Contracts

We mostly follow the notation of Hatfield and Milgrom (2005). There are finite sets D and H of doctors and hospitals, and a finite set X of contracts. Each contract $x \in X$ is associated with one doctor $x_D \in D$ and one hospital $x_H \in H$. Each doctor $d \in D$ can sign at most one contract and his null contract where he signs no contract is denoted by $\emptyset_d$. A set of contracts $X' \subseteq X$ is an **allocation** if each doctor is associated with at most one contract under X'.

For each doctor $d \in D$, $\succ_d$ is a strict preference relation on his contracts $X_d \equiv \{x \in X |\, x_D = d\} \cup \{\emptyset_d\}$. A contract is **acceptable** by doctor d if it is at least as good as the null contract $\emptyset_d$, and **unacceptable** by doctor d if it is worse that the null contract $\emptyset_d$. For each doctor $d \in D$ and a set of contracts $X' \subseteq X$, the **chosen set** $C_d(X')$ of doctor d is defined as

$$C_d(X') \equiv \max_{\succ_d} \Big(\{x \in X' |\, x_D = d\} \cup \{\emptyset_d\} \Big).$$

For a given set of contracts $X' \subseteq X$, define $C_D(X') \equiv \bigcup_{d \in D} C_d(X')$, and the **rejected set** of doctors as $R_D(X') \equiv X' - C_D(X')$.

Given a hospital $h \in H$ and any set of contracts $X' \subseteq X$, define $X'_h \equiv \{x \in X' |\, x_H = h\}$ to be the set of its non-empty contracts in X'. Given a set of contracts $X' \subseteq X$, the **chosen set** $C_h(X')$ of an hospital h is a subset of the contracts associated with hospital h. That is, $C_h(X') \subseteq X'_h$. Moreover, a hospital can sign only one contract with any given doctor:

$$\forall h \in H, \forall X' \subseteq X, \forall x, x' \in C_h(X') \qquad x \neq x' \implies x_D \neq x'_D.$$

For a given set of contracts $X' \subseteq X$, define $C_H(X') \equiv \bigcup_{h \in H} C_h(X')$, and the **rejected set** of hospitals as $R_H(X') \equiv X' - C_H(X')$.

For a given hospital $h \in H$, we refer the function that maps each set of contracts to a chosen set as the **choice function** of hospital h.

In Hatfield and Milgrom (2005), it is not entirely clear whether hospital choice functions are primitives of the model, or it is the hospital preferences that are primitives and these preferences are used to construct hospital choice functions. In page 917, Hatfield and Milgrom (2005) state:

> The choices of a hospital h are more complicated, because it has preferences $\succ_h$ over sets of doctors.

This quote suggests that they may have a model in mind where hospital preferences are the primitives and choice functions are derived from these primitives. But if so, what are the implications of that modeling choice on the structure of choice functions? More precisely, choice sets will need to comply with the *strong axiom of revealed preference* (SARP) to avoid cycles in underlying preferences. What are the implications of SARP on choice functions in this domain? Or, how are the choice sets determined if there are indifferences in hospital preferences? None of these important questions are answered in the paper, if hospital preference relations are intended as primitives. (In contrast, doctor choice sets are clearly defined in the paper based on strict doctor preferences over contracts.)

Since the entire analysis in Hatfield and Milgrom (2005) treats hospital choice functions as if they were the primitives (with the important exception of a number of proofs), we will for now consider hospital choice functions to be primitives of the model.[3] This distinction is very important for our observations, and we will return to this important discussion in Section 6.

3 Stability under the Substitutes Condition: A Counter Example

Stability axiom plays a central role in analysis of two-sided matching models and it is extended to matching with contracts as follows.

Definition 1 *A set of contracts $X' \subseteq X$ is a* ***stable allocation*** *(or a* ***stable set of contracts****) if*

1. $C_D(X') = C_H(X') = X'$, *and*

2. *there exists no hospital $h \in H$ and set of contracts $X'' \neq C_h(X')$ such that*

$$X'' = C_h(X' \cup X'') \subseteq C_D(X' \cup X'').$$

When the first condition fails, the allocation X' fails **individual rationality** and there is a blocking doctor or a hospital. When the second condition fails, there is a blocking coalition that consists of an hospital h and a subset of doctors $\{x_D\}_{x \in X''}$. In this case we say that X'' **blocks** X' .

Hatfield and Milgrom (2005) claim that the set of of stable allocations is non-empty under the following condition:

[3]Treatment of choice functions to be primitives is the modeling choice of early many-to-many matching models such as Blair (1988).

Definition 2 *Contracts are* ***substitutes*** *for hospital h if there do not exist a set of contracts $Y \subset X$ and a pair of contracts $x, z \in X \setminus Y$ such that*

$$z \notin C_h(Y \cup \{z\}) \text{ and } z \in C_h(Y \cup \{x, z\}).$$

Loosely speaking, the substitutes condition captures the intuitive idea that a contract that is rejected from a set of contracts shall remain to be rejected when there is "increased competition." The following example shows that the set of stable allocations may be empty under the substitutes condition, in the absence of additional structure:

Example 1 *Consider a problem with one hospital h, and two doctors d_1, d_2. Doctor d_1 has two contracts x, x' and doctor d_2 has one contract y. Preferences of the doctors and the choice function of the hospital are given as follows:*

$$\succ_{d_1}: \quad x \succ_{d_1} x' \succ_{d_1} \emptyset_{d_1}$$

$$\succ_{d_2}: \quad y \succ_{d_2} \emptyset_{d_2}$$

$C_h(\{x\}) = \{x\}$	$C_h(\{x, x'\}) = \{x\}$	$C_h(\{x, x', y\}) = \emptyset$
$C_h(\{x'\}) = \{x'\}$	$C_h(\{x, y\}) = \{y\}$	
$C_h(\{y\}) = \{y\}$	$C_h(\{x', y\}) = \{x'\}$	

It is easy to verify that C_h satisfies the substitutes condition. Moreover no allocation is stable in this example. Here is a list of blocking coalitions for each possible allocation:

Allocation	*Blocking Coalition*	*Allocation*	*Blocking Coalition*
$\{x\}$	$\{h, d_2\}$ *via* y	$\{x, y\}$	$\{h\}$ *via removing* x
$\{x'\}$	$\{h, d_1\}$ *via* x	$\{x', y\}$	$\{h\}$ *via removing* y
$\{y\}$	$\{h, d_1\}$ *via* x'	$\emptyset$	$\{h, d_1\}$ *via* x

The existence claim of Hatfield and Milgrom (2005) is not only key for several of their results, but also for a large number of follow-up papers on matching with contracts. The source of the complication is the following technical claim in Hatfield and Milgrom (2005), their Theorem 1, which together with fix-point techniques from lattice theory are heavily used to obtain strong implications on the structure of stable allocations.

Claim 1 *If $(X_D, X_H) \subseteq X^2$ is a solution to the system of equations*

$$X_D = X - R_H(X_H) \quad \text{and} \quad X_H = X - R_D(X_D), \tag{1}$$

then $X_H \cap X_D$ is a stable set of contracts and $X_H \cap X_D = C_D(X_D) = C_H(X_H)$. Conversely, for any stable collection of contracts X', there exists some pair (X_D, X_H) satisfying (1) such that $X' = X_H \cap X_D$.

Our next example shows that both parts of Claim 1 fail to hold in the absence of additional structure.

Example 2 *Consider a problem with one hospital h, and two doctors d_1, d_2. Doctor d_1 has two contracts x, x' and doctor d_2 has two contracts y, y'. Preferences of doctors and the choice function of hospital are given as follows:*

$$\succ_{d_1}: \quad x \succ_{d_1} x' \succ_{d_1} \emptyset_{d_1}$$
$$\succ_{d_2}: \quad y' \succ_{d_2} y \succ_{d_2} \emptyset_{d_2}$$

$C_h(\{x\}) = \{x\}$	$C_h(\{x,x'\}) = \{x\}$	$C_h(\{x,x',y\}) = \emptyset$	$C_h(\{x,x',y,y'\}) = \emptyset$
$C_h(\{x'\}) = \{x'\}$	$C_h(\{x,y\}) = \{x,y\}$	$C_h(\{x,x',y'\}) = \emptyset$	
$C_h(\{y\}) = \{y\}$	$C_h(\{x,y'\}) = \{y'\}$	$C_h(\{x,y,y'\}) = \emptyset$	
$C_h(\{y'\}) = \{y'\}$	$C_h(\{x',y\}) = \{x'\}$	$C_h(\{x',y,y'\}) = \emptyset$	
	$C_h(\{x',y'\}) = \{x',y'\}$		
	$C_h(\{y,y'\}) = \{y\}$		

It is easy to verify that there are two stable allocations in this problem: $\{x,y\}$ and $\{x',y'\}$. However there exists no pair (X_D, X_H) which satisfies the system of equations (1) with $X_H \cap X_D \in \big\{\{x,y\},\{x',y'\}\big\}$. Indeed the only solution to the system of equations (1) is $(X_D, X_H) = \big(\emptyset, \{x,x',y,y'\}\big)$, but in this case $X_D \cap X_H = \emptyset$ is not stable since, for example, every singleton blocks it.

Observe that while contracts are not assumed to be substitutes in the statement of Claim (1), they satisfy this condition in our example. Therefore in addition to refuting Claim (1), this example also shows that the set of stable allocations may fail the lattice structure even when there are stable allocations.

4 A Remedy with the Irrelevance of Rejected Contracts

A close look at the proof of Theorem 1 in Hatfield and Milgrom (2005) reveals the source of the complication. The following additional condition on hospital choice functions is implicitly assumed throughout the paper.

Definition 3 *Contracts satisfy the* ***irrelevance of rejected contracts (IRC)*** *for hospital h if*

$$\forall Y \subset X, \forall z \in X \setminus Y \qquad z \notin C_h(Y \cup \{z\}) \implies C_h(Y) = C_h(Y \cup \{z\}).$$

That is, the removal of rejected contracts shall not affect the choice set.[4]

Remark 1 *IRC is implicitly assumed in the proof of Theorem 1 in Hatfield and Milgrom (2005) and it is informally referred as "revealed preference for hospitals." While a revealed preference argument works for doctors since their choice sets are constructed via their preference relations, it may fail to hold for hospitals when hospital choice functions are primitives.*

As far as we can see, IRC restores the results of Hatfield and Milgrom (2005).[5] Instead of going through the crucial role of IRC in several of their results, we simply show that the existence result is recovered once IRC is assumed in addition to substitutes. We will use a constructive proof showing that the following **cumulative offer algorithm** (Hatfield and Milgrom 2005) always gives a stable allocation.

Step 1: One of the doctors offers her first choice contract x_1. The hospital receiving the offer, $h_1 = (x_1)_H$, holds the contract if $x_1 \in C_{h_1}\big(\{x_1\}\big)$ and rejects it otherwise. Let $A_{h_1}(1) = \{x_1\}$, and $A_h(1) = \emptyset$ for all $h \in H \setminus \{h_1\}$.

In general, at

Step t: One of the doctors with no contract on hold offers her most preferred contract x_t that has not been rejected in earlier steps. The hospital receiving the offer, $h_t = (x_t)_H$, holds the contracts in $C_{h_t}\big(A_{h_t}(t-1) \cup \{x_t\}\big)$ and rejects the rest. Let $A_{h_t}(t) = A_{h_t}(t-1) \cup \{x_t\}$, and $A_h(t) = A_h(t-1)$ for all $h \in H \setminus \{h_t\}$.

The algorithm terminates when either every doctor is matched to a hospital or every unmatched doctor has had all acceptable contracts rejected. Since each contract is offered at most once, the algorithm terminates in some finite Step T. The outcome of the algorithm is, $\bigcup_{h \in H} C_h\big(A_h(T)\big)$.

Theorem 1 *Suppose contracts satisfy the substitutes condition and the IRC condition. Then the set of stable allocations is non-empty.*

Proof. We will prove the theorem by showing that the outcome of the cumulative offer algorithm is stable. First observe that a rejected contract is never held in the steps following its rejection by the substitutes condition. Therefore the algorithm will always terminate and will yield an allocation. Let T be the step the algorithm terminates, and X' be its outcome. For any doctor d, let x'_d be the contract of doctor d under allocation X'. Note that this contract can be the null contract $\emptyset_d$, in case doctor d has no contract in X'.

[4]This condition is earlier used by Blair (1988) in the context of many-to-many matching. In an extension of Blair's results, Alkan (2002) refers it as *consistency*.

[5]An exception is Theorem 5, which was earlier shown to be erroneous in Hatfield and Kojima (2008).

We will next show that X' is stable. First observe that no doctor can block X' since a doctor never offers an unacceptable contract. Hence $C_D(X') = X'$. Next suppose $C_H(X') \neq X'$, and observe that $C_H(X') = \bigcup_{h \in H} C_h\big(A_h(T)\big)$ under IRC. Therefore there exists a hospital h and a contract x such that $x \in C_h(A_h(T))$ but $x \notin C_h\big(C_h(A_h(T))\big)$. Since $C_h(A_h(T)) \subseteq A_h(T)$, this is ruled out by the substitutes condition. Hence $C_H(X') = X'$.

Finally, towards a contradiction, suppose there exists a hospital h and a set of contracts $X'' \neq C_h(X')$ such that

$$X'' = C_h(X' \cup X'') \subseteq C_D(X' \cup X'').$$

Let $X'_h = \{x \in X' | \ x_H = h\}$. That is, X'_h is the subset of X' that pertains to hospital h. Observe that $X'_h = C_h\big(A_h(T)\big)$ by the mechanics of the cumulative offer algorithm. Also recall that, we have already shown $C_h(X') = X'_h$ by the above individual rationality argument. Hence

$$X'_h = C_h(X') = C_h\big(A_h(T)\big). \tag{2}$$

Since $X'' = C_h(X' \cup X'')$, we have $x_H = h$ for all $x \in X''$. Moreover since $X'' \subseteq C_D(X' \cup X'')$,

$$\forall x \in X'' \qquad x \ \succeq_{x_D} \ x'_{x_D}.$$

Therefore each contract in X'' is offered to hospital h by step T by the mechanics of the cumulative offer algorithm. Hence

$$X'' \subseteq A_h(T). \tag{3}$$

This in turn implies

$$X'' = C_h(X' \cup X'') = C_h(X'_h \cup X'') = C_h\big(C_h(X') \cup X''\big) = C_h(A_h(T)) = C_h(X')$$

contradicting $X'' \neq C_h(X')$. Here

1. the first equality holds by assumption;
2. the second equality holds by IRC since none of the contracts in $X' \setminus X'_h$ pertain to hospital h, and as such they are automatically rejected by h;
3. the third equality holds by the Relation (2);
4. the fourth equality holds by IRC together with Relations (2) and (3) since $\big(C_h(X') \cup X''\big) \subseteq A_h(T)$ and only the rejected contracts are removed between $A_h(T)$ and $\big(C_h(X') \cup X''\big)$; and
5. the last equality holds by the Relation (2).

This shows that X' is stable completing the proof. ■

Observe that, IRC is heavily used in the proof.

5 Relation with the Law of Aggregate Demand

Much of the literature on matching with contracts, including several results in Hatfield and Milgrom (2005), assumes the following condition in addition to the substitutes condition.

Definition 4 *Contracts satisfy the **law of aggregate demand (LAD)** for hospital h if,*

$$\forall X', X'' \subseteq X \qquad X' \subset X'' \implies |C_h(X')| \leq |C_h(X'')|.$$

A bit of a good news is that substitutes along with LAD implies IRC, and hence results in the literature assuming LAD are immune to our criticism.

Proposition 1 *Suppose contracts satisfy the substitutes condition along with the LAD condition for hospital h. Then contracts also satisfy the IRC condition for hospital h.*

Proof. Suppose contracts satisfy the substitutes condition along with the LAD condition for hospital h. Let $Y \subset X$ and $z \in X \setminus Y$ be such that $z \notin C_h(Y \cup \{z\})$. We want to show that $C_h(Y) = C_h(Y \cup \{z\})$.

For any $x \in C_h(Y \cup \{z\})$, we have $x \neq z$ by assumption. This implies $x \in Y$ which in turn implies $x \in C_h(Y)$ by the substitutes condition. Therefore

$$C_h(Y \cup \{z\}) \subseteq C_h(Y).$$

Moreover we have $|C_h(Y)| \leq |C_h(Y) \cup \{z\}|$ by the LAD and hence the above inclusion must hold with equality completing the proof. ■

6 Hospital Choices: Primitives or Endogenous?

As we have seen, IRC is needed together with the substitutes condition throughout Hatfield and Milgrom (2005) if hospital choice functions are to be treated as primitives of the model. An alternative possibility is deriving the hospital choice functions from given hospital preference relations. Indeed this is the approach taken by Hatfield and Kojima (2010). In their version of the model, each hospital $h \in H$ has preference relation $\succ_h$ on

$$\Big\{Y \subseteq X_h |\ y, y' \in Y \text{ and } y \neq y' \implies y_D \neq y'_D\Big\},$$

and for any $X' \subseteq X$, the **chosen set** $C_h(X')$ of hospital h is defined as

$$C_h(X') \equiv \max_{\succ_h} \Big\{Y \subseteq X'_h |\ y, y' \in Y \text{ and } y \neq y' \implies y_D \neq y'_D\Big\}.$$

While doctor preferences are explicitly assumed to be strict under Hatfield and Kojima (2010), no such assumption is made on hospital preferences. Unless hospital preferences are also required to be strict, the above definition of a chosen set might be problematic, as a hospital might be indifferent between several of its top choices. In this case one would need to choose single-valued selections from hospitals' choice *correspondences*. But our counter examples are still valid under this approach unless IRC is imposed on these selections. To see this point, suppose a hospital is indifferent between all feasible sets of contracts. In this case, any choice function is consistent with these preferences, including those presented in Examples 1 and 2.

The last possibility is deriving hospital choice functions from given hospital preferences as in Hatfield and Kojima (2010), but further assuming that the underlying preferences are strict. It is easy to see that contracts satisfy IRC under this approach.[6] That is, hospital choice functions can no longer be assumed to be fully flexible under this scenario. Observe, however, that, IRC is not the only condition that needs to hold under this modeling choice. For example generalized versions of the cycles observed under the following example shall be ruled out, as their presence would contradict transitivity of the underlying hospital preferences:

Example 3 *There is one hospital h and three doctors with one acceptable contract each. The three contracts are x, y, z and hospital choice function is given by the following table:*

$C_h(\{x\}) = \{x\}$	$C_h(\{x, y\}) = \{x\}$	$C_h(\{x, y, z\}) = \{x, y, z\}$
$C_h(\{y\}) = \{y\}$	$C_h(\{x, z\}) = \{z\}$	
$C_h(\{z\}) = \{z\}$	$C_h(\{y, z\}) = \{y\}$	

Let $\succ_h$ be the underlying preference relation. Since $x \succ_h y$ by $C_h(\{x, y\}) = \{x\}$, $y \succ_h z$ by $C_h(\{y, z\}) = \{y\}$, and $z \succ_h x$ by $C_h(\{x, z\}) = \{z\}$, the strong axiom of revealed preference is violated, and hence the above choice function is not consistent with a strict preference order.

More formally, hospital choice functions shall satisfy the following property in order to respect the transitivity of the underlying preferences.

Definition 5 *Contracts satisfy the* ***Strong Axiom of Revealed Preference (SARP)*** *for hospital h, if there exists no distinct $X^1, X^2, \ldots, X^k \subset X$ and distinct $Y^1, Y^2, \ldots, Y^k \subset X$ with $k > 1$, such that*

$$\forall \ell \in \{1, \ldots, k\} \qquad Y^\ell = C_h(X^\ell), \text{ and}$$
$$\forall \ell \in \{1, \ldots, k-1\} \quad Y^\ell \subset X^\ell \cap X^{\ell+1} \qquad \text{and} \qquad Y^k \subset X^k \cap X^1.$$

[6]See, for example, Lemma 1 in Hatfield, Immorlica and Kominers (2012) for a short proof of this observation.

As far as we can see, SARP is not needed in any of the proofs of Hatfield and Milgrom (2005). Hence assuming that hospital choice functions are derivatives of the underlying strict hospital preferences not only imposes a strong structure on "feasible" choice functions on which the substitutes condition must be superimposed,[7] but also potentially weakens the scope of their analysis. As such, assuming hospital choice functions to be primitives of the model and restating the results by explicitly assuming IRC might be the preferred approach. If, however, one takes strict hospital preference relations to be primitives, it is important to understand how SARP interacts with the substitutes condition. Here an important observation is, as in the case of the IRC condition, SARP might be violated even when contracts satisfy the substitutes condition. This is indeed the case in Example 1. However, as we show next, the substitutes condition together with IRC implies SARP.

Proposition 2 *Suppose contracts satisfy the substitutes condition along with the IRC condition for hospital h. Then contracts also satisfy SARP for hospital h.*

Proof. Suppose contracts satisfy the substitutes condition along with the IRC condition for hospital h. Towards a contradiction, suppose SARP is violated. Then there exists distinct $X^1, X^2, \ldots, X^k \subset X$ and distinct $Y^1, Y^2, \ldots, Y^k \subset X$ with $k > 1$, such that

$$\begin{array}{lll} \forall \ell \in \{1, \ldots, k\} & Y^\ell = C_h(X^\ell), \text{ and} & \\ \forall \ell \in \{1, \ldots, k-1\} & Y^\ell \subset X^\ell \cap X^{\ell+1} \qquad \text{and} & \qquad Y^k \subset X^k \cap X^1. \end{array}$$

Define $\overline{X} = \bigcup_{\ell \le k} X^\ell$, $\overline{Y} = \bigcup_{\ell \le k} Y^\ell$, and $\underline{Y} = \bigcap_{\ell \le k} Y^\ell$. Also define $Y^{k+1} \equiv Y^1$ and $X^{k+1} \equiv X^1$ for notational convenience. For any $x \in X$,

$$x \in \overline{X} \setminus \overline{Y} \implies \exists \ell \le k \text{ s.t. } x \in X^\ell \setminus Y^\ell \implies x \notin C_h(\overline{X}) \tag{4}$$

where the last implication holds by the substitutes condition. Moreover for any $x \in X$,

$$x \in \overline{Y} \setminus \underline{Y} \implies \exists \ell \le k \text{ s.t. } x \in Y^\ell \setminus Y^{\ell+1} \implies x \in X^{\ell+1} \setminus Y^{\ell+1} \implies x \notin C_h(\overline{X}) \tag{5}$$

where the second implication holds by the relation $Y^\ell \subset X^\ell \cap X^{\ell+1} \subset X^{\ell+1}$ and the third implication holds by the substitutes condition. Therefore by (4) and (5) we have, for any $x \in X$,

$$x \in \overline{X} \setminus \underline{Y} \implies x \notin C_h(\overline{X}) \implies C_h(\overline{X}) \subseteq \underline{Y}. \tag{6}$$

[7] Understanding the role of the substitutes condition in the results under this modeling choice would be analogous to trying to appreciate a picture which is drawn on top of another picture.

Pick any $\ell \leq k$. We have $\overline{X} \supseteq X^\ell \supseteq Y^\ell \supset \underline{Y} \supseteq C_h(\overline{X})$ where the last inclusion holds by (6). Therefore for any $\ell \leq k$, we must have $Y^\ell = C_h(X^\ell) = C_h(\overline{X})$ by IRC contradicting the distinct choice of sets $Y^1, Y^2, \ldots, Y^k$ and completing the proof. ■

An immediate corollary of Propositions 1 and 2 is the following.

Corollary 1 *Suppose contracts satisfy the substitutes condition along with the LAD condition for hospital h. Then contracts also satisfy SARP for hospital h.*

7 Concluding Remarks

Our observations have potentially adverse implications on a large number of follow-up papers on matching with contracts. However two strands of the literature are mostly shielded from our criticism. A significant portion of the literature assume LAD in addition to the substitutes condition. Substitutes along with LAD implies IRC and hence our criticism has no bite under LAD. In addition, market design applications of matching with contracts, including the earlier mentioned applications on school choice with soft caps and cadet-branch matching, typically construct choice sets based on other primitives including but not limited to preferences, thereby automatically satisfy the IRC condition. Therefore most of the results in market design applications are likely correct, even though their proofs might be slightly inaccurate.

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